Stabilisation H_∞

de systèmes fractionnaires à retards

de type neutre

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Outline of the talk

1. Background on general (fractional) delay systems

2. An extension of the Walton-Marshall technique (location of zeros)

3. H_∞-stability characterization of a class of fractional delay systems

4. H_∞-control characterization of a class of fractional delay systems

Linear Differential Equation with Discrete Delays

$$\begin{cases} \dot{x}(t) + \sum_{i=0}^{k} E_i \dot{x}(t-\mu_i) & = \sum_{i=0}^{k} A_i x(t-t_i) + \\ & \qquad \sum_{i=0}^{m} B_i u(t-\tau_i) \\ y(t) & = \sum_{i=0}^{l} C_i x(t-\sigma_i) + \\ & \qquad \sum_{i=0}^{p} d_i u(t-\nu_i) \end{cases}$$

where $x(t) \in \mathbb{R}^n$, $u(t)$, $y(t) \ \in \mathbb{R}$,
A_i, B_i, C_i are $n \times n$, $n \times 1$ and $1 \times n$ matrices
and $d_i \ \in \mathbb{R}$.

In the Laplace domain

$$
\begin{aligned}
G(s) = & \left(\sum_{i=0}^{m} B_i e^{-s\tau_i}\right) \\
\times & \left(sI - s\sum_{i=0}^{k} E_i e^{-s\mu_i} + \sum_{i=0}^{k} A_i e^{-st_i}\right)^{-1} \\
\times & \left(\sum_{i=0}^{l} C_i e^{-s\sigma_i}\right) + \sum_{i=0}^{p} d_i e^{-s\nu_i}.
\end{aligned}
$$

$$
G(s) = \frac{\displaystyle\sum_{i=0}^{n_2} q_i(s) e^{-\beta_i s}}{\displaystyle\sum_{i=0}^{n_1} p_i(s) e^{-\gamma_i s}}
$$

$0 = \gamma_0 < \gamma_1 \cdots < \gamma_{n_1}$, $0 \leq \beta_0 < \beta_1 \cdots < \beta_{n_2}$

the p_i, q_i are polynomials

Location of poles (Bellman & Cooke, 1963)

Condition 1: deg $p_0 >$ deg p_i for $i = 1, \cdots, n_1$ and deg $p_0 \geq$ deg q_i for $i = 0, \cdots, n_2$.

Retarded type

G has a finite number of poles in any right half-plane

Condition 2: deg $p_0 \geq$ deg p_i for $i = 1, \cdots, n_1$ (with equality for at least one polynomial p_i) and deg $p_0 \geq$ deg q_i for $i = 0, \cdots, n_2$.

Neutral type

The poles of G are in a band around the imaginary axis.

Fractional delay systems

$$G(s) = \frac{\sum_{i=0}^{n_2} q_i(s) e^{-\beta_i s}}{\sum_{i=0}^{n_1} p_i(s) e^{-\gamma_i s}}$$

the p_i are polynomials of the form

$\sum_{k=0}^{l_i} a_k s^{\alpha_k}$ with $\alpha_k \in \mathbb{R}_+$

$s \in \mathbb{C} \setminus \mathbb{R}_-$, $-\pi < \arg s < \pi$.

retarded type, neutral type

example (Matignon) $G(s) = \frac{1}{(a\sqrt{s}+b)+(c\sqrt{s}+d)e^{-sh}}$
with $a \neq 0$, b, $c \neq 0$, d reals

H_∞-stability of such systems

Condition 1

G is H_∞-stable (BIBO-stable, nuclear) iff G has no poles in the cut right-half plane.

Condition 2

If there exists $a < 0$ such that P has no poles in $(\mathbb{C} \setminus \mathbb{R}_-) \cap \{\mathrm{Re}\, s > a\} \cup \{0\}$ then P is H_∞-stable (BIBO-stable).

This condition is not necessary.

In the case of commensurate delays, we can garantee that there is only a finite number of poles in a “large half-plane”.

The class of systems studied

$$G(s) = \frac{r(s)}{p(s) + q(s)e^{-sh}}$$

$h > 0$,

p,q,r real polynomials in the variable s^{μ} for $0 < \mu < 1$.

$\deg p = \deg q$ and $\deg r \leq \deg p$.

Here the degree is interpreted as the degree in s^{μ} and so is an integer.

Extension of the Walton-Marshall method

Determining the values of $h \geq 0$ for which a function $A(s) + C(s)e^{-sh}$ with A, C real polynomials, has all its zeroes in the left half-plane.

- Test stability at $h = 0$.
- Test stability for infinitesimally small $h > 0$.
- Locate the zero-crossings and classify them as stabilizing or destabilizing. Calculate the corresponding values of h and associated stability windows.

Extension of the Walton-Marshall method

If $A(s) + C(s)e^{-sh} = 0$ has a zero at the point $s = i\omega \in i\mathbb{R} \setminus \{0\}$, then

$$W(\omega) := |A(i\omega)|^2 - |C(i\omega)|^2 = 0.$$

Moreover, at such a point s, provided that A and C do not vanish, one has

$$\operatorname{sign} \operatorname{Re} \frac{ds}{dh} = \operatorname{sign} \operatorname{Re} \frac{1}{s} \left[\frac{C'(s)}{C(s)} - \frac{A'(s)}{A(s)} \right]$$

H_∞-stability of $G(s) = \dfrac{r(s)}{p(s) + q(s)e^{-sh}}$

Let $\alpha = \lim_{\substack{s \in \mathbb{C} \setminus R_- \\ |s| \to \infty}} p(s)/q(s)$,

If $|\alpha| < 1$ then G has infinitely many unstable poles asymptotic to a vertical line $\operatorname{Re} s = -\log|\alpha|/h$ in the right half-plane and thus G cannot lie in H_∞.

If $|\alpha| > 1$ then the poles of G of large modulus are asymptotic to a vertical line strictly in the left half-plane; thus G has at most finitely many unstable poles, and if there are none, then G lies in H_∞.

Suppose $\deg r < \deg p$. If $|\alpha| < 1$ then G cannot be stabilized by a (proper) rational controller.

Behaviour of large poles : the case $\alpha = 1$

Suppose that

$$\frac{p(s)}{q(s)} = \alpha + \frac{\beta}{s^{\mu}} + \frac{\gamma}{s^{2\mu}} + \frac{\delta}{s^{3\mu}} + O(s^{-4\mu}) \qquad \text{as} \quad |s| \to \infty,$$

for constants α,β,γ and δ with $\alpha = \pm 1$.

(i) If $\beta/\alpha > 0$, then the large poles of G are stable, and if $\beta/\alpha < 0$, then they are unstable.

(ii) If $\beta = 0$ and $0 < \mu < \frac{1}{2}$ then the large poles are

- unstable for $\gamma/\alpha < 0$
- stable for $\gamma/\alpha > 0$;

For $\beta = 0$ and $\frac{1}{2} < \mu < 1$, then the large poles are

- stable for $\gamma/\alpha < 0$
- unstable for $\gamma/\alpha > 0$.

(iii) If $\beta = 0$ and $\mu = \frac{1}{2}$, then the large poles are

- unstable if $\delta/\alpha > 0$
- stable if $\delta/\alpha < 0$.

Complete characterization of H_∞-stability

If the poles of G are all in the open left half-plane, then the transfer function G lies in H_∞

- in case (i) if and only if $\deg p \geq \deg r + 1$,

- in case (ii) if and only if $\deg p \geq \deg r + 2$,

- in case (iii) if and only if $\deg p \geq \deg r + 3$.

H_∞-stabilization of $G(s) = \dfrac{1}{as^\mu + b + (cs^\mu + d)e^{-sh}}$

- We determine first a particular controller:

$K(s) = k_p + \dfrac{k_i}{s^\mu}$

- We choose a controller which stabilizes G at $h = 0$, for h small, for all h.

- Remark: A quadratic equation $a_2 z^2 + a_1 z + a_0$ with $a_2 > 0$, a_1, a_0 reals has roots in $\{z \in \mathbb{C}, -\dfrac{\pi}{2}\mu \leq \arg z \leq \dfrac{\pi}{2}\mu\}$ only if one of the next three conditions holds:

i) $a_0 > 0$, $a_1^2 - 4a_2a_0 < 0$, $a_1 < 0$ and $\sqrt{|a_1^2 - 4a_2a_0|} \leq (-a_1)\tan\frac{\pi}{2}\mu$, in which case there are two complex conjugate roots

ii) $a_0 < 0$, in which case there is one positive real root

iii) $a_0 > 0$, $a_1^2 - 4a_2a_0 > 0$ and $a_1 < 0$ in which case there are two positive real roots.

1) If $|\frac{a}{c}| > 1$ then every fractional PI controller K which stabilizes G when $h = 0$ will also stabilize G when h is sufficiently small. Moreover, if $k_i > 0$ and k_p satisfy also $(a(b + k_p) - cd)\cos\frac{\pi}{2}\mu > 0$, $(b + k_p)^2 + 2ak_i\cos\pi\mu - d^2 > 0$ and $k_i(b + k_p)\cos\frac{\pi}{2}\mu > 0$, then K will stabilize G for all h.

2) Let $a = c$. If k_p and k_i satisfy none of the following conditions:

(i) $(b + d + k_p)^2 - 8k_i < 0$, $b + d + k_p < 0$ and $\sqrt{|(b + d + k_p)^2 - 8k_i|} \leq -(b + d + k_p)\tan\frac{\pi}{2}\mu$

(ii) $(b + d + k_p)^2 - 8k_i > 0$ and $b + d + k_p < 0$

then K stabilizes G when $h = 0$.

Moreover if $a(b + k_p - d)\cos\frac{\pi}{2}\mu > 0$, then K stabilizes G for small h.

If k_p and k_i satisfy also $(b + k_p)^2 + 2ak_i\cos\pi\mu - d^2 > 0$ and $k_i(b + k_p)\cos\frac{\pi}{2}\mu > 0$ then K stabilizes G for all h.

Let $G(s) = \dfrac{1}{as^\mu + b + (cs^\mu + d)e^{-sh}}$

Suppose that $|a| > |c|$; then the set of all H_∞-stabilizing controllers is given by $\dfrac{V + MQ}{U - NQ}$, where

$$
\begin{aligned}
N(s) &= \frac{1}{s^\mu + 1}, \quad M(s) = \frac{(as^\mu + b) + (cs^\mu + d)e^{-sh}}{s^\mu + 1}, \\
U(s) &= \frac{s^\mu(s^\mu + 1)}{((as^\mu + b) + (cs^\mu + d)e^{-sh})s^\mu + k_p s^\mu + k_i}, \\
V(s) &= \frac{(s^\mu + 1)(k_i + k_p s^\mu)}{((as^\mu + b) + (cs^\mu + d)e^{-sh})s^\mu + k_p s^\mu + k_i},
\end{aligned}
$$

Q is a free parameter in H_∞ and $k_i > 0$ and k_p satisfy $(a(b + k_p) - cd) \cos \frac{\pi}{2}\mu > 0$,$(b + k_p)^2 + 2ak_i \cos \pi\mu - d^2 > 0$, and $k_i(b + k_p) \cos \frac{\pi}{2}\mu > 0$

Example: let $G(s) = \dfrac{1}{2\sqrt{s} - (\sqrt{s}+1)e^{-sh}}$

If $k_i > 0$ and k_p satisfy the following inequalities

$$(2k_p - 1)\cos\frac{\pi}{4} > 0,\ k_p^2 - 1 > 0 \quad \text{and} \quad k_i k_p \cos\frac{\pi}{4} > 0,$$

that is, if $k_i > 0$ and $k_p > 1$,
then the closed-loop is stable for all h.

A coprime factorization of G is given by

$$(N,M) = \left(\frac{1}{\sqrt{s}+1}, \frac{2\sqrt{s} - (\sqrt{s}+1)e^{-sh}}{\sqrt{s}+1}\right)$$

The set of all H_∞-stabilizing controllers of G is given by $\dfrac{V + MQ}{U - NQ}$ where

$$U(s) = \frac{\sqrt{s}(\sqrt{s}+1)}{(2\sqrt{s} - (\sqrt{s}+1)e^{-sh})\sqrt{s} + k_p\sqrt{s} + k_i}$$

$$V(s) = \frac{(\sqrt{s}+1)(k_i + k_p\sqrt{s})}{(2\sqrt{s} - (\sqrt{s}+1)e^{-sh})\sqrt{s} + k_p\sqrt{s} + k_i},$$

and Q is a free parameter in H_∞.