H^∞ Feedback Boundary Stabilization of 2D Navier-Stokes Equations

Sheetal Dharmatti[1] J.P. Raymond [1]

[1]Laboratoire MIP
Université Paul Sabatier, Toulouse

Réunion EDP-MOSAR, 22-23 March 2007, Paris

Boundary stabilization problem for nonlinear NSE

Aim: To determine a Dirichlet boundary feedback control u robust with respect to disturbance w, so that

$$|y(t) - v|_{\mathbf{X}} \to 0 \quad \text{as } t \to \infty$$

where

$$\begin{aligned}&\frac{\partial y}{\partial t} - \nu\Delta y + (y\cdot\nabla)v + (v\cdot\nabla)y + (y\cdot\nabla)y + \nabla p = 0,\\ &\text{div } y = 0 \ \text{ in } Q_\infty,\ y(0) = y_0 \text{ in } \Omega\\ &y = Mu \ \text{ on } \Gamma_i \times (0,\infty),\ y = w \ \text{ on } \Gamma_e \times (0,\infty)\end{aligned} \tag{1}$$

and v is regular, unstable solution of the stationary NSE,

$$\begin{aligned}&-\nu\Delta v + (v\cdot\nabla)v + \nabla\chi = f\\ &\text{div } v = 0 \text{ in } \Omega,\ v = u_s^\infty \text{ on } \Gamma_i \text{ and } v = w_s^\infty \text{ on } \Gamma_e.\end{aligned} \tag{2}$$

Linearized Boundary Stabilization Problem

To study the local feedback stabilization of system (1), we study the feedback stabilization of the corresponding linearized system

$$\frac{\partial y}{\partial t} - \nu \Delta y + (v \cdot \nabla) y + (y \cdot \nabla) v + \nabla p = 0, \quad \text{div } y = 0 \ \text{ in } Q_\infty,$$
$$y = Mu \text{ on } \Gamma_i \times (0,\infty), \ y = w \text{ on } \Gamma_e \times (0,\infty), \ y(0) = y_0 \text{ in } \Omega. \qquad (3)$$

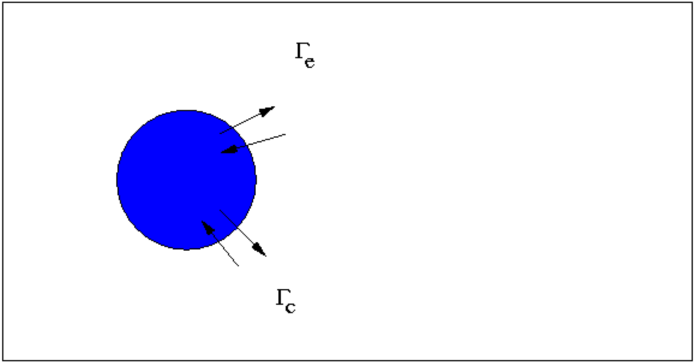
and show that the solution of the closed loop system with feedback law decays exponentially for y_0 and w small enough.

Questions

The important questions to be addressed are:

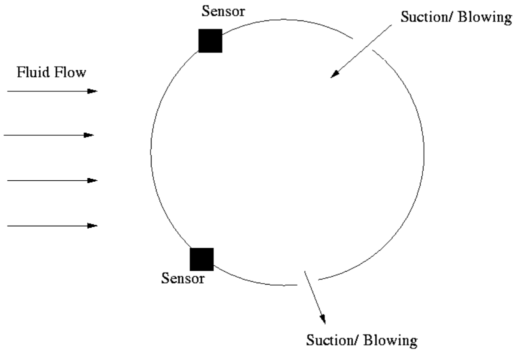
- Is the linearized system (3) stabilizable with point wise feedback law ?
- Can we find an equation characterizing K which can be solved by classical methods ?
- If K is a point wise feedback law able to stabilize system (3). Does K also stabilize the nonlinear system (1) for $|y_0|$ and $|w|$ small enough ?

Localized boundary controls



u_s a constant longitudinal velocity is given on Γ_e, w a disturbance on Γ_e, Γ_c is a part of the cylinder boundary

Estimation + feedback control



Plan of the talk

- Review of previous work
- Define H^{∞} optimal control problem for linearized system
- To obtain point wise feedback law by solving infinite horizon problem
  - To solve finite horizon problem and get corresponding Differential Riccati Equation (DRE)
  - To pass to the limit and to get Algebraic Riccati equation (ARE)
- To show that feedback law stabilizes corresponding nonlinear system

Plan of the talk

- Review of previous work
- Define H^{∞} optimal control problem for linearized system
- To obtain point wise feedback law by solving infinite horizon problem
  - To solve finite horizon problem and get corresponding Differential Riccati Equation (DRE)
  - To pass to the limit and to get Algebraic Riccati equation (ARE)
- To show that feedback law stabilizes corresponding nonlinear system

Plan of the talk

- Review of previous work
- Define H^∞ optimal control problem for linearized system
- To obtain point wise feedback law by solving infinite horizon problem
    - To solve finite horizon problem and get corresponding Differential Riccati Equation (DRE)
    - To pass to the limit and to get Algebraic Riccati equation (ARE)
- To show that feedback law stabilizes corresponding nonlinear system

Plan of the talk

- Review of previous work
- Define H^{∞} optimal control problem for linearized system
- To obtain point wise feedback law by solving infinite horizon problem
  - To solve finite horizon problem and get corresponding Differential Riccati Equation (DRE)
  - To pass to the limit and to get Algebraic Riccati equation (ARE)
- To show that feedback law stabilizes corresponding nonlinear system

Plan of the talk

- Review of previous work
- Define H^{∞} optimal control problem for linearized system
- To obtain point wise feedback law by solving infinite horizon problem
    - To solve finite horizon problem and get corresponding Differential Riccati Equation (DRE)
    - To pass to the limit and to get Algebraic Riccati equation (ARE)
- To show that feedback law stabilizes corresponding nonlinear system

Plan of the talk

- Review of previous work
- Define H^{∞} optimal control problem for linearized system
- To obtain point wise feedback law by solving infinite horizon problem
    - To solve finite horizon problem and get corresponding Differential Riccati Equation (DRE)
    - To pass to the limit and to get Algebraic Riccati equation (ARE)
- To show that feedback law stabilizes corresponding nonlinear system

Previous work

- Stabilization of optimal control of NSE
    - Fursikov, Gunzburger, Hou
    - Barbu, Lasieka, Triggiani
    - J. P. Raymond
- H^∞ control problems
    - Bensoussan, Bernhard
    - Barbu, Sritharan

Previous work

- Stabilization of optimal control of NSE
    - Fursikov, Gunzburger, Hou
    - Barbu, Lasieka, Triggiani
    - J. P. Raymond
- H^{∞} control problems
    - Bensoussan, Bernhard
    - Barbu, Sritharan

Previous work

- Stabilization of optimal control of NSE
    - Fursikov, Gunzburger, Hou
    - Barbu, Lasieka, Triggiani
    - J. P. Raymond
- H^∞ control problems
    - Bensoussan, Bernhard
    - Barbu, Sritharan

Previous work

- Stabilization of optimal control of NSE
    - Fursikov, Gunzburger, Hou
    - Barbu, Lasieka, Triggiani
    - J. P. Raymond
- H^∞ control problems
    - Bensoussan, Bernhard
    - Barbu, Sritharan

Previous work

- Stabilization of optimal control of NSE
    - Fursikov, Gunzburger, Hou
    - Barbu, Lasieka, Triggiani
    - J. P. Raymond
- H^∞ control problems
    - Bensoussan, Bernhard
    - Barbu, Sritharan

H^∞Optimal Control Problem

$$\sup_{w} \inf_{u} \{ J_T(s, y, u, w) \;\mid\; (y, u, w) \text{satisfies } (4) \}, \qquad (\mathcal{P}^T_{s,\zeta})$$

where

$$J_T(s, y, u, w) = \frac{1}{2} \int_s^T \int_\Omega |y|^2 + \frac{1}{2} \int_s^T \int_{\Gamma_i} |u(t)|^2 - \frac{\gamma}{2} \int_s^T \int_{\Gamma_e} |w(t)|^2,$$

$$\begin{aligned} &\frac{\partial y}{\partial t} - \nu \Delta y + (v \cdot \nabla) y + (y \cdot \nabla) v + \nabla p = 0, \text{ in } Q_{(s,T)}, \text{ div } y = 0 \\ &y = Mu \text{ on } \Gamma_i \times (s, T), \; y = w \text{ on } \Gamma_e \times (s, T), \; y(s) = \zeta \text{ in } \Omega. \end{aligned} \qquad (4)$$

Functional framework

Let us introduce the following spaces:

$$H^s(\Omega;\mathbb{R}^N) = \mathbf{H}^s(\Omega)\ ,\ L^2(\Omega;\mathbb{R}^N) = \mathbf{L}^2(\Omega)$$

Divergence free function spaces and corresponding trace spaces:

$$\mathbf{V}^s(\Omega) = \Big\{ y \in \mathbf{H}^s(\Omega) \mid \ \text{div } y = 0 \ \text{ in } \Omega,\ \langle y\cdot n, 1\rangle_{H^{-1/2}(\Gamma),H^{1/2}(\Gamma)} = 0 \Big\},$$

$$\mathbf{V}^s_n(\Omega) = \Big\{ y \in \mathbf{H}^s(\Omega) \mid \ \text{div } y = 0 \ \text{ in } \Omega,\ \mathbf{y}\cdot n = 0 \text{ on } \Gamma \Big\} \text{ for } s \geq 0,$$

$$\mathbf{V}^s_0(\Omega) = \Big\{ y \in \mathbf{H}^s(\Omega) \mid \ \text{div } y = 0 \ \text{ in } \Omega,\ y = 0 \text{ on } \Gamma \Big\} \text{ for } s > 1/2,$$

$$\mathbf{V}^s(\Gamma) = \Big\{ y \in \mathbf{H}^s(\Gamma) \mid \langle y\cdot n, 1\rangle_{H^{-1/2}(\Gamma),H^{1/2}(\Gamma)} = 0 \Big\} \text{ for } s \geq -1/2\,.$$

Stokes and Oseen's operators

Let P be the orthogonal projector in $\mathbf{L}^2(\Omega)$ onto $\mathbf{V}^0_n(\Omega)$, and denote by $(A_0, D(A_0))$, $(A, D(A))$ and $(A^*, D(A^*))$ the unbounded operators in $\mathbf{V}^0_n(\Omega)$ defined by

- Oseen operator

$$D(A_0) = \mathbf{H}^2(\Omega) \cap \mathbf{V}^1_0(\Omega),\ A_0 y = P\Delta y \ \forall\ y \in D(A_0),$$

- Stokes operator

$$D(A) = \mathbf{H}^2(\Omega) \cap \mathbf{V}^1_0(\Omega),\ Ay = \nu P\Delta y - P((v\cdot\nabla)y) - P((y\cdot\nabla)v),$$

- Adjoint of Stokes operator

$$D(A^*) = \mathbf{H}^2(\Omega) \cap \mathbf{V}^1_0(\Omega),\ A^* y = \nu P\Delta y + P((v\cdot\nabla)y) - P((\nabla v)^T y).$$

Dirichlet operators

- $D_c g = y$ and $D_p = q$ where (y, q) satisfies

$$\begin{aligned} \lambda_0 y - \nu \Delta y + (v \cdot \nabla) y + (y \cdot \nabla) v + \nabla q = 0, \ \text{ in } \Omega, \\ \text{div } y = 0 \text{ in } \ \Omega, \ y = g \text{ on } \ \Gamma_i, \end{aligned}$$

- Similarly for disturbance we define, $D_d h = y$ and $D_p = q$ where (y, q) satisfies

$$\begin{aligned} \lambda_0 y - \nu \Delta y + (v \cdot \nabla) y + (y \cdot \nabla) v + \nabla q = 0, \ \text{ in } \Omega, \\ \text{div } y = 0 \text{ in } \ \Omega, \ y = h \text{ on } \ \Gamma_e \end{aligned}$$

Moreover we define operators:

$$B_c g = (\lambda_0 I - A) P D_c g \quad B_d h = (\lambda_0 I - A) P D_d h$$

Dirichlet operators

- $D_c g = y$ and $D_p = q$ where (y, q) satisfies

$$\lambda_0 y - \nu \Delta y + (v \cdot \nabla) y + (y \cdot \nabla) v + \nabla q = 0, \quad \text{in } \Omega,$$
$$\text{div } y = 0 \text{ in } \Omega, \; y = g \text{ on } \Gamma_i,$$

- Similarly for disturbance we define, $D_d h = y$ and $D_p = q$ where (y, q) satisfies

$$\lambda_0 y - \nu \Delta y + (v \cdot \nabla) y + (y \cdot \nabla) v + \nabla q = 0, \quad \text{in } \Omega,$$
$$\text{div } y = 0 \text{ in } \Omega, \; y = h \text{ on } \Gamma_e$$

Moreover we define operators:

$$B_c g = (\lambda_0 I - A) P D_c g \quad B_d h = (\lambda_0 I - A) P D_d h$$

Dirichlet operators

- $D_c g = y$ and $D_p = q$ where (y, q) satisfies

$$\begin{aligned} \lambda_0 y - \nu \Delta y + (v \cdot \nabla) y + (y \cdot \nabla) v + \nabla q = 0, \;\; \text{in } \Omega, \\ \text{div } y = 0 \text{ in } \;\Omega, \;\; y = g \text{ on } \;\Gamma_i, \end{aligned}$$

- Similarly for disturbance we define, $D_d h = y$ and $D_p = q$ where (y, q) satisfies

$$\begin{aligned} \lambda_0 y - \nu \Delta y + (v \cdot \nabla) y + (y \cdot \nabla) v + \nabla q = 0, \;\; \text{in } \Omega, \\ \text{div } y = 0 \text{ in } \;\Omega, \;\; y = h \text{ on } \;\Gamma_e \end{aligned}$$

Moreover we define operators:

$$B_c g = (\lambda_0 I - A) P D_c g \quad B_d h = (\lambda_0 I - A) P D_d h$$

Properties of operators

- $\lambda_0 > 0$ is an element in the resolvent set of A satisfying

$$((\lambda_0 I - A)y, y)_{\mathbf{V}^0_n(\Omega)} \geq \omega_0 |y|^2_{\mathbf{V}^1_0(\Omega)} \ \forall y \in D(A),$$

$$((\lambda_0 I - A^*)y, y)_{\mathbf{V}^0_n(\Omega)} \geq \omega_0 |y|^2_{\mathbf{V}^1_0(\Omega)} \ \forall y \in D(A^*),$$

  for some $0 < \omega_0 < 1$.

- The operator D_c and D_d are bounded operators from $\mathbf{V}^0(\Gamma)$ into $\mathbf{V}^0(\Omega)$, and

$$\|D_c u\|_{\mathbf{V}^{s+1/2}(\Omega)} \leq C(s)\|u\|_{\mathbf{V}^s(\Omega)} \ \forall \ 0 \leq s \leq 2$$

- B_c and B_d belong to $\mathcal{L}(\mathbf{V}^0(\Gamma), \mathbf{V}^0_n(\Omega))$.

Rewriting linearized NSE

We can rewrite equation (4) in the form

$$\begin{aligned} Py' &= APy + B_c Mu + B_d w \quad \text{in } (0,T),\ y(0) = y_0, \\ (I-P)y &= (I-P)D_c Mu + (I-P)D_d w \quad \text{in } (0,T). \end{aligned}$$

and solve

$$\sup_w \inf_u \left\{ J_T(0,y,u,w) \middle| \, u \in L^2((0,T)\times \mathbf{V}^0(\Gamma_i)), w \in L^2((0,T)\times \mathbf{V}^0(\Gamma_e)) \right\} \qquad (\mathcal{P}^T_{0,y_0})$$

Results for finite time horizon

Theorem

There exists a critical value $\gamma_T > 0$ *such that for all* $\gamma > \gamma_T$ *the problem* $(\mathcal{P}^T_{0,y_0})$ *admits a unique solution* (y_T, u_T, w_T) *given by:*

$$u_T = -MB_c^*\Phi_T, \quad w_T = \frac{1}{\gamma}B_d^*\Phi_T$$

where (y_T, Φ_T) *is a unique solution of the system,*

$$\begin{aligned} y' &= Ay + B_c Mu + B_d w \text{ in } (0,T),\ y(0) = y_0, \\ -\Phi' &= A^*\Phi + y \quad \text{in } (0,T),\ \Phi(T) = 0. \end{aligned} \tag{5}$$

Results for finite time horizon

Theorem

Conversely for $\gamma > \gamma_T$, the system

$$Py' = APy - B_c M^2 B_c^* \Phi + \frac{1}{\gamma} B_d B_d^* \Phi \text{ in } (0, T),\ P\mathbf{y}(0) = y_0 \qquad (6)$$
$$-\Phi' = A^* \Phi + Py \quad \text{in } (0, T),\ \Phi(T) = 0,$$

admits a unique solution (Py_T, Φ_T) in $L^2(0, T; \mathbf{V}_n^0(\Omega)) \times (\mathbf{V}^{2,1}(Q_{s,T})) \cap L^2(0, T; \mathbf{V}_0^1(\Omega))$.

Characterization of γ_T

$$\gamma_T = \sup_{\|w\|_{L^2((0,T);\mathbf{V}^0(\Gamma_e))}=1} \int_0^T \int_{\Gamma_e} \langle w, \Lambda_T w \rangle,$$

where

$$\Lambda_T w = B_d^* \Psi_T$$

and for fixed w, (z_T, Ψ_T) is a unique solution of

$$\begin{aligned} Pz' &= APz + B_c M^2 B_c^* \Psi + B_d w \text{ in } (0,T),\ z(0) = 0, \\ -\Psi' &= A^* \Psi + Pz \quad \text{in } (0,T),\ \Psi(T) = 0. \end{aligned} \tag{7}$$

Results for finite time horizon

Theorem

$$\begin{aligned} Py' &= APy + B_cMu + B_dw \text{ in } (s,T),\ y(s) = \zeta, \\ -\Phi' &= A^*\Phi + Py \quad \text{in } (s,T),\ \Phi(T) = 0. \end{aligned} \tag{8}$$

The unique solution of above system, written as $(y^T_{s,\zeta}, \Phi^T_{s,\zeta}) \in \mathbf{V}^{1,1/2}(Q_{s,T}) \times L^2(s,T;\mathbf{V}^3(\Omega)) \cap \mathbf{V}^1_0(\Omega) \cap H^{3/2}(s,T;\mathbf{V}^0_n(\Omega))$. *The optimal solution for the problem* $(\mathcal{P}^T_{s,\zeta})$ *is given by* $(y^T_{s,\zeta}, u^T_{s,\zeta} = -MB^*_c\Phi^T_{s,\zeta}, w^T_{s,\zeta} = \frac{1}{\gamma}B^*_d\Phi^T_{s,\zeta})$ *Moreover* $\Phi^T_{s,\zeta} \in C([s,T];\mathbf{V}^2(\Omega) \cap \mathbf{V}^1_0(\Omega))$

Feedback Operator Π and DRE

Let $\Pi(s)$ be the operator defined by

$$\Pi(s) \; : \; \zeta \longmapsto \mathbf{\Phi}^s_\zeta(s) \tag{9}$$

where $(Py^s_\zeta, \Phi^s_\zeta)$ is the unique solution of the system (8).

Riccati Equation: It can be shown that Π is the unique solution in $C_s([0,T];\mathcal{L}(\mathbf{V}^0_n(\Omega)))$ to the Riccati equation

$$-\Pi'(t) = A^*\Pi(t) + \Pi(t)A - \Pi(t)B_cM^2B_c^*\Pi(t) - \tfrac{1}{\gamma}\Pi(t)B_dB_d^*\Pi(t) + I,$$

$$\Pi(T) = 0, \quad \Pi^*(t) = \Pi(t) \text{ and } \quad \Pi(t) \geq 0, \tag{10}$$

Feedback Operator Π and DRE

Let $\Pi(s)$ be the operator defined by

$$\Pi(s)\ :\ \zeta \longmapsto \mathbf{\Phi}_\zeta^s(s) \tag{9}$$

where $(Py_\zeta^s, \Phi_\zeta^s)$ is the unique solution of the system (8).

Riccati Equation: It can be shown that Π is the unique solution in $C_s([0,T];\mathcal{L}(\mathbf{V}_n^0(\Omega)))$ to the Riccati equation

$$-\Pi'(t) = A^*\Pi(t) + \Pi(t)A - \Pi(t)B_cM^2B_c^*\Pi(t) - \frac{1}{\gamma}\Pi(t)B_dB_d^*\Pi(t) + I,$$

$$\Pi(T) = 0, \quad \Pi^*(t) = \Pi(t) \text{ and } \quad \Pi(t) \geq 0, \tag{10}$$

Problem Description

$$\sup_{w} \inf_{u} \{ J(y,u,w) \ | \ (y,u,w) \text{satisfies } (11) \}, \qquad (\mathcal{P}_{\infty})$$

where

$$J(y,u,w) = \frac{1}{2} \int_0^{\infty} \int_{\Omega} |y|^2 + \frac{1}{2} \int_0^{\infty} \int_{\Gamma_i} |u(t)|^2 - \frac{\gamma}{2} \int_0^{\infty} \int_{\Gamma_e} |w(t)|^2$$

and,

$$Py' = APy + B_c Mu + B_d w \text{ in } \ (0,\infty), \ y(0) = y_0, \qquad (11)$$

Main difficulties

- Is the minimization problem well posed ?
  For all $w \in L^2(0, \infty; L^2(\Gamma_e))$ and $y_0 \in V_n^0(\Omega)$ there exists control u such that $y_{y_0,u,w} \in L^2(Q_\infty)$.
- Passing to the limit from finite horizon to infinite horizon!
- The sup inf problem is well posed the solution is obtained in the feed back form for optimal u and w.
  But we cannot take optimal w in general as w is not known!!!

Well posedness of minimization problem

For

$$Py' = APy + B_c Mu + B_d w \quad \text{in } (0,\infty), \quad y(0) = y_0 \tag{12}$$

for fixed w solve the the minimization problem

$$\inf_u \left\{ I(y,u,\beta) \;\middle|\; (y,u,w) \text{ satisfies } (12),\; u \in \mathbf{V}^{0,0}(\Sigma_\infty), \; \right\},$$

where $I(y,u,\beta) =$

$$\frac{1}{2}\int_0^\infty \int_\Omega |-(A_0)^{1/2} Py|^2 + \frac{1}{2}\int_0^\infty \int_{\Gamma_i} |u(t)|^2$$

This is well posed and so existence of a control for each w is guaranteed!

Passage to the limit

Let y_k, u_k, w_k be the optimal triplet in the interval $(0, k)$ with initial condition y_0.

- Use previous known results to deduce that for fixed w, y_k and u_k are weakly convergent and depend linearly on w.
- Show that γ_k are monotonically increasing.
- Since Λ_k is bounded for each k, γ_k are bounded above.
- Define the limit of Λ_k to be Λ_∞ and

$$\gamma_\infty = \sup_{\|w\|_{L^2((0,\infty);\mathbf{V}^0(\Gamma_e))}=1} \int_0^\infty \int_{\Gamma_e} \langle w, \Lambda_\infty w \rangle$$

Results for infinite time horizon

Theorem

There exists a critical $\gamma_\infty > 0$ *such that for all* $\gamma > \gamma_\infty$ *and for all* $y_0 \in V_n^0(\Omega)$ *the problem* $(\mathcal{P}_\infty)$ *admits a unique solution* $(y_\infty, u_\infty, w_\infty)$ *where* (y_∞, Φ_∞) *is the unique solution of the system,*

$$y' = Ay + B_c Mu + B_d w \text{ in } (0,\infty),\ y(0) = y_0,$$

$$-\Phi' = A^*\Phi + y \quad \Phi(\infty) = 0.$$

and u_∞, w_∞ *are characterized by*

$$u_\infty = -MB_c^*\Phi_\infty, \quad w_\infty = \frac{1}{\gamma} B_d^* \Phi_\infty$$

Algebraic Riccati equation

We define feedback operator Π by

$$\Pi y(t) = \Phi(t) \quad \forall t \in (0, \infty)$$

It can be shown that Π satisfies algebraic Riccati equation given by:

$$\begin{aligned}
&A^*\Pi + \Pi A - \Pi B_c M^2 B_c^* \Pi + \tfrac{1}{\gamma}\Pi B_d B_d^* \Pi + I = 0,\\
&\Pi^* = \Pi \in \mathcal{L}(\mathbf{V}_n^0(\Omega)) \quad \text{and} \quad \Pi \geq 0,\\
&\text{for all } \mathbf{y} \in \mathbf{V}_n^0(\Omega),\ \Pi\mathbf{y} \in \mathbf{V}^2(\Omega) \cap \mathbf{V}_0^1(\Omega) \quad \text{and}\\
&|\Pi\mathbf{y}|_{\mathbf{V}^2(\Omega)} \leq C|\mathbf{y}|_{\mathbf{V}_n^0(\Omega)}
\end{aligned}$$

Theorem

The unbounded operator $(A_\Pi, D(A_\Pi))$ *defined by:*

$$D(A_\Pi) = \left\{ y \in \mathbf{V}_n^0(\Omega) \mid A - BM^2B^*\Pi + \frac{1}{\gamma}B_dB_d^*\Pi \in \mathbf{V}_n^0(\Omega) \right\},$$

$$A_\Pi y = Ay - B_cM^2B_c^*\Pi y + \frac{1}{\gamma}B_dB_d^*\Pi y \quad \textit{for all } y \in D(A_\Pi),$$

is the infinitesimal generator of an analytic exponentially stable semi group on $\mathbf{V}_n^0(\Omega)$.

Result for arbitrary disturbance w

Consider Navier-Stokes equations with the linear feedback law Π:

$$\begin{aligned}
&Py' = Ay - B_c M^2 B_c^* \Pi y + B_d w - P(y\cdot\nabla)y \quad \text{in } (0,\infty),\\
&Py(0) = y_0,\\
&(I-P)y = -(I-P)D_c M^2 B_c^* \Pi P y + (I-P)D_d w.
\end{aligned} \tag{13}$$

Theorem

For all $w \in V^{1/2,1/4}$, there exists $\mu_0 > 0$ and a nondecreasing function $\eta : \mathbb{R}^+ \to \mathbb{R}^+$, such that for $\mu \in (0,\mu_0)$ and $|y_0|_{\mathbf{V}^0_n(\Omega)} + \|w\|_{V^{1/2,1/4}(\Sigma_e)} \leq \eta(\mu)$, the equation (13) admits a unique solution in the set

$$D_\mu = \left\{ y \in \mathbf{V}^{1,1/4}(Q_\infty) \mid \|y\|_{\mathbf{V}^{1,1/4}(Q_\infty)} \leq \mu \right\}.$$

Outline of the proof

Proof by fixed point method:
For $z \in \mathbf{V}^{1,1/4}(\Omega)$, we denote by y_z the solution to the equation

$$\begin{aligned} &Py' = Ay - B_c M^2 B_c^* \Pi y + B_d w + PF(z), \qquad y(0) = \mathbf{y}_0, \\ &(I-P)y = -(I-P)D_c M R_A^{-1} M B_c^* \Pi P y + (I-P) D_d B_d w \end{aligned} \tag{14}$$

We prove that the mapping $M \ : \ z \mapsto y_z$ is a contraction in D_μ. ☐

Thank You!