Asymptotic stability of the Webster-Lokshin model

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Abstract

The aim of this paper is to study the asymptotic stability of some wave equation with fractional damping accounting for visco-thermal losses at the walls of a flared pipe; moreover, the radiation boundary condition at the end of the pipe is described by a positive real impedance.

The difficulty of this model is twofold: first the fractional differential operator is non-local in time and must be transformed into a diffusive realization in the sense of systems theory; second, although a global energy can be built for this system, made of the wave energy and the diffusive energy, LaSalle's invariance principle does not apply, since a lack of compactness is to be found in this model. In this case, a refined analysis of the spectrum of the generator of the semigroup is needed, in order to apply Arendt–Batty stability theorem. This has already been carried out on the ODE corresponding to the projection on only one mode in [Matignon & Prieur, ESAIM: COCV, 2005], but the question is even more difficult to tackle on the whole PDE.

Keywords: diffusive representations, well-posed systems, Lyapunov functional, fractional calculus, lack of compactness, asymptotic stability.

Mathematics Subject Classification. 93C20, 93D20, 35B37

1 INTRODUCTION

Our goal is to study the internal asymptotic stability of an infinite-dimensional linear model, namely a wave equation in a 1-D bounded domain. A classical undamped wave equation is known to be a conservative systems, which can be described by a group of operators. On our more realistic model, there are two physical causes of dissipation: the damping at the boundaries and the internal damping.

First note that usual boundary conditions at the two ends of the pipe, either Dirichlet or Neumann boundary conditions are reflecting and account for a conservation of the wave energy; on the contrary, boundary conditions of *impedance* type are absorbing, and translate into dissipation of the wave energy, localized at the boundaries only. Most models of impedance are formulated in the frequency domain, and not the time domain; hence, since the impedances at stake, seen as transfer functions, happen to be positive real, one can apply the Kalman-Yakubovich-Popov lemma to build a realization, at least in finite dimension. The latter realization happens to be of major help in deriving an energy balance, which will prove useful in the stability analysis of the coupled system.

Second, there are different types of internal damping models for waves, corresponding to losses during the propagation; the most common ones are *fluid* or *viscous* damping, and *Kelvin-Voigt* damping. Both these models are local in time, and allow for a straightforward semigroup formulation. Fluid damping corresponds to a uniform shift of the poles in the spectral domain, or to an exponential window in the time domain: the stability analysis of the system is quite elementary, see e.g. [14, theorem 5.38]. With *Kelvin-Voigt* damping, the high-frequency modes are more heavily damped than the low-frequency ones, a situation which does occur in applications, making this model more realistic; the stability analysis can be performed by various methods, see e.g. [14, section 4.3].

We are now concerned with a more complex damping model, known as damping of fractional order in time (not to be confused with fractional damping in space, which involves fractional powers of the positive operator $-\Delta$): causal fractional integrals or derivatives are non-local operators in time, which require an infinite-dimensional diagonal realization of diffusive type in order to get a semigroup formulation. An energy inequality is associated to this formulation, and a natural way to proceed to analyze stability would then be to use LaSalle's invariance principle; in infinite dimension, this principle requires to check the precompactness of the trajectories in the extended energy space. Unfortunately there is no simple way to check this property, since the diffusive realization is made on an unbounded domain. This is the reason why we resort to some more sophisticated stability theorem, which requires the analysis of the spectrum of the generator of the extended semigroup.

The necessary ingredients, namely mathematical tools, will be introduced step by step, in order to understand where true and sometimes hidden complexity lies in our hierarchy of damped wave models. The paper is organized as follows: in section 2, we consider damping models which enable the use of LaSalle's invariance principle, thanks to a compactness property. Some of them are quite simple and do not require any realization theory: fluid damping, absorbing boundary conditions with a *constant* impedance; they are just recalled in § 2.1. Some other are more complex, and require some realization theory in finite dimension: fluid or structural damping, absorbing boundary conditions with a positive-real impedance given by a rational function; they are presented in § 2.2. Finally in section 3, we consider complex models, which require some realization theory in infinite dimension, and for which no compactness property can be found, thus forbidding the use of LaSalle's invariance principle; for such models, a refined analysis of the spectrum of the generator of the semigroup is carried out: reflecting or absorbing boundary conditions, pseudo-differential damping of diffusive type, such as fractional derivatives or integrals. Section 4 is devoted to conclusions on the problems treated in this paper and future works, including straightforward generalizations and interesting open questions.

2 DAMPING MODELS WITH A COMPACTNESS PROPERTY

2.1 Simple models without realization theory

We consider a wave equation $\partial_t^2 \phi - \frac{1}{r^2}\partial_z(r^2\,\partial_z\phi) = 0$, where $z \in [0,\,1]$ is the space variable and the *varying* radius of the duct fulfills $r \in L^\infty(0,1;\mathbb{R}^+)$ and $r \geq r_0 > 0$.

2.1.1 Constant impedances at the boundaries

Working on first order systems in the $(p,\,v)$ variables, with $p = \partial_t\phi$ and $v = -r^2\,\partial_z\phi$, the model rewrites:

$$\begin{aligned} \partial_t p &= -r^{-2}\,\partial_z v\ , & (1)\\ \partial_t v &= -r^2\,\partial_z p\ , & (2)\\ p_i(t) &= \mp \mathcal{Z}_i\, v_i(t) \quad \text{for } i = 0,1\ . & (3) \end{aligned}$$

The static boundary conditions (3) at $z = i$ are formulated in the time domain, with shorthand notation $p_i(t) = p(z = i,\, t)$; the impedances fulfill $0 \leq \mathcal{Z}_i \leq +\infty$, $\mathcal{Z}_0 + \mathcal{Z}_1 > 0$ and ${\mathcal{Z}_0}^{-1} + {\mathcal{Z}_1}^{-1} > 0$.

As far as well-posedness is concerned, this model can be easily analyzed, using semigroup theory. The functional spaces to be used are:

$$L^2_p = \{p,\int_0^1 p^2 r^2(z)\,dz < +\infty\},\ H^1_p = \{p \in L^2_p, \int_0^1 [p^2 + (\partial_z p)^2] r^2(z) dz < +\infty\},$$
$$L^2_v = \{v,\int_0^1 v^2 r^{-2}(z)\,dz < +\infty\},\ H^1_v = \{v \in L^2_v, \int_0^1 [v^2 + (\partial_z v)^2] r^{-2}(z) dz < +\infty\}\ ,$$

then $\mathcal{H} = L^2_p \times L^2_v$ as state space, and $\mathcal{V} = H^1_p \times H^1_v$ to define the domain of operator $\mathcal{A}$ as $D(\mathcal{A}) = \left\{ (p,v)^T \in \mathcal{V},\ \middle|\ \begin{array}{l} p(z=0) = -\mathcal{Z}_0\, v(z=0) \\ p(z=1) = \mathcal{Z}_1\, v(z=1) \end{array} \right\}$. The *monotonicity* of $\mathcal{A}$ follows from the identity: $(\mathcal{A}X,\, X)_\mathcal{H} = \mathcal{Z}_0\,|v(z=0)|^2 + \mathcal{Z}_1\,|v(z=1)|^2 \geq 0$, $\forall X \in D(\mathcal{A})$. The proof of *maximality* of $\mathcal{A}$ can be done as in [10], it is a little bit involved, due to the boundary conditions.

Finally, *asymptotic stability* can be easily proved using LaSalle's invariance principle: the only equilibrium point is $(p^*, v^*) = (0,0)$. Now the resolvent of $\mathcal{A}$ is compact, because $D(\mathcal{A})$ is a closed subspace of $\mathcal{V}$ and the embedding of $\mathcal{V}$ into $\mathcal{H}$ is compact thanks to Rellich theorem; we then conclude with [14, Theorem 3.65].

Note that the modes of this system form a Riesz basis of the energy space $\mathcal{H}$ (see e.g. [11] for a full proof, which makes the equivalent scalar product explicit).

2.1.2 Fluid damping

Studying the wave equation $\partial_t^2\phi + \varepsilon(z)\,\partial_t\phi - \frac{1}{r^2}\partial_z(r^2\,\partial_z\phi) = 0$ with $\varepsilon \in L^\infty(0,1;\mathbb{R}^+)$ does not require more realization theory either: only the operator $\mathcal{A}$ is modified, not the functional spaces. So, $\mathcal{A}$ generates a $\mathcal{C}^0$-semigroup of contraction on $\mathcal{H}$, see e.g. [5]. And even for *reflecting* boundary conditions (namely $\mathcal{Z}_i = 0$ or $+\infty$ for $i = 0, 1$), it can be proved that the system is asymptotically stable, provided that $\varepsilon(z) \geq \varepsilon_0 > 0$;

otherwise, some intervals in $[0, 1]$ can be left undamped. However, care must be taken that the equilibrium points are $(p^*, v^*) = (0, v^*)$ in the four possible *reflecting* cases.

2.2 Models with realization theory in finite dimension

When more realistic frequency-dependent boundary conditions are considered, one needs to check their positivity and then rewrite them as a dynamical system with positivity properties, see e.g. [4]. The stability of a coupled PDE-ODE system has also been studied in [6], using LaSalle's invariance principle. The coupling between passive subsystems is being used as main method of analysis, as in [14, ch. 5].

Let us consider the first order system (1)–(2) in the $(p,\, v)$ variables with:

$$\hat{p}_i(s) = \mp \mathcal{Z}_i(s)\, \hat{v}_i(s) \quad \text{for } i = 0, 1\ . \tag{4}$$

The dynamical boundary conditions (4) at $z = i$ are formulated in the Laplace domain, with shorthand notation $p_i(t) = p(z = i,\, t)$; the impedances $\mathcal{Z}_i(s)$ are strictly positive real, i.e. $\Re e(\mathcal{Z}_i(s)) > 0$, $\forall s$, $\Re e(s) \geq 0$.

2.2.1 Dissipative realizations for positive-real impedances (Kalman-Yakubovich-Popov lemma)

For a strictly positive real impedance $\mathcal{Z}_i(s)$ of rational type, we choose a *minimal* realization $(A_i,\, B_i,\, C_i,\, d_i)$ with state x_i of *finite* dimension n_i ($A_i \in \mathbb{R}^{n_i \times n_i}$, $B_i \in \mathbb{R}^{n_i \times 1}$, $C_i \in \mathbb{R}^{1 \times n_i}$ and $d_i \in \mathbb{R}$); then, following e.g. [23], there exists $P_i \in \mathbb{R}^{n_i \times n_i}$, $P_i = P_i^T > 0$, such that the following energy balance holds:

$$\int_0^T p_i(t)\, v_i(t)\, dt = \frac{1}{2}(x_i^T(T)\, P_i\, x_i(T)) + \frac{1}{2}\int_0^T \begin{pmatrix} x_i^T(t) & v_i(t) \end{pmatrix} \mathcal{M}_i \begin{pmatrix} x_i(t) \\ v_i(t) \end{pmatrix} dt\ ,$$

with $\mathcal{M}_i = \begin{pmatrix} -A_i^T P_i - P_i A_i & C_i^T - P_i B_i \\ C_i - B_i^T P_i & 2 d_i \end{pmatrix} \geq 0$.

2.2.2 An abstract formulation

Thus, the global system (1)–(2)–(4) can be put in the abstract form $\partial_t X + \mathcal{A}\, X = 0$, where:

$$\mathcal{A} \begin{pmatrix} x_0 \\ x_1 \\ p \\ v \end{pmatrix} = \begin{pmatrix} -A_0 x_0 - B_0\, v(z=0) \\ -A_1 x_1 - B_1\, v(z=1) \\ r^{-2}\, \partial_z v \\ r^2\, \partial_z p \end{pmatrix}\ ; \tag{5}$$

together with the boundary conditions $p(z = 0) = -C_0 x_0 - d_0\, v(z = 0)$ and $p(z = 1) = C_1 x_1 + d_1\, v(z = 1)$. In order to analyze the well-posedness of this system, we define the state space $\mathcal{H} = \mathbb{R}^{n_0} \times \mathbb{R}^{n_1} \times L^2_p \times L^2_v$ and, with $\mathcal{V} = \mathbb{R}^{n_0} \times \mathbb{R}^{n_1} \times H^1_p \times H^1_v$, the domain of operator $\mathcal{A}$ as:

$$D(\mathcal{A}) = \left\{ (x_0, x_1, p, v)^T \in \mathcal{V}, \;\middle|\; \begin{array}{l} p(z=0) = -C_0 x_0 - d_0\, v(z=0) \\ p(z=1) = C_1 x_1 + d_1\, v(z=1) \end{array} \right\}\ .$$

The *monotonicity* of $\mathcal{A}$ follows from the identity: $\forall X \in D(\mathcal{A})$,

$$(\mathcal{A}X,\, X)_{\mathcal{H}} = \frac{1}{2}\begin{pmatrix} x_0^T & v(0)\end{pmatrix}\mathcal{M}_0\begin{pmatrix} x_0 \\ v(0)\end{pmatrix} + \frac{1}{2}\begin{pmatrix} x_1^T & v(1)\end{pmatrix}\mathcal{M}_1\begin{pmatrix} x_1 \\ v(1)\end{pmatrix} \geq 0\,. \tag{6}$$

The details of the proof of *maximality* of $\mathcal{A}$ can be found in [10].

At this stage, it must be noticed that $\mathcal{A}$ also has *compact* resolvent, because the new components, x_0 and x_1, are finite-dimensional. So, there is no problem to apply LaSalle's invariance principle to prove that the only equilibrium point is the origin $(x_0^*, x_1^*, p^*, v^*) = 0$, at least in the fairly general case $\mathcal{M}_0 > 0$ and $\mathcal{M}_1 > 0$. Hence the system is asymptotically stable.

3 DAMPING MODELS WITH NO COMPACTNESS PROPERTY

The model under study, found originally in [12, 13], reads:

$$\partial_t^2 \phi + \eta(z)\,\partial_t^{3/2}\phi + \varepsilon(z)\,\partial_t^{1/2}\phi - \frac{1}{r^2}\partial_z(r^2\,\partial_z\phi) = 0\,; \tag{7}$$

static boundary conditions are associated to (7). The striking spectral and time-domain consequences of such an integro-differential perturbation of the wave PDE can be found in e.g. [19].

Working on first order systems in the $(p,\, v)$ variables leads to:

$$\partial_t p = -r^{-2}\,\partial_z v - \varepsilon\,\partial_t^{-1/2} p - \eta\,\partial_t^{1/2} p\,, \tag{8}$$

$$\partial_t v = -r^2\,\partial_z p\,, \tag{9}$$

$$p(z=0,t) = 0 \quad \text{and} \quad v(z=1,t) = 0\,. \tag{10}$$

Here we first need to realize both pseudo-differential operators, namely $\partial_t^{-1/2}$ as a standard diffusive realization in § 3.1.1, and $\partial_t^{1/2}$ as an extended diffusive realization in § 3.1.2. Then, in § 3.2, we rewrite (8)-(9) as a coupled system with four state variables: this will help prove the well-posedness of the system using semigroup theory.

3.1 Diffusive realizations in infinite dimension

Diffusive realizations of causal pseudo-differential operators have been introduceg e.g. in [24, § 5], with many counterparts and extensions: the matrix-valued case has been investigated in [22] with an application to the beam; the formulation in discrete time has been studied in [7]. In the sequel, we present the standard diffusive realization which applies to fractional integrals, and the extended diffusive realization which applies to fractional derivatives.

3.1.1 Standard diffusive realizations

Let M a positive measure on $\mathbb{R}^+$ satisfying the well-posedness condition $c_M = \int_0^\infty \frac{dM}{1+\xi} < +\infty$. Consider the dynamical system with input $u \in L^2(0,T)$, output $y \in L^2(0,T)$

and state $\phi \in H_M = L^2(\mathbb{R}^+, dM)$:

$$\partial_t \phi(\xi,t) \;=\; -\xi\,\phi(\xi,t) \,+\, u(t); \quad \phi(\xi,0) \,=\, 0, \quad \forall\, \xi \in \mathbb{R}^+ \;, \tag{11}$$

$$y(t) \;=\; \int_0^{+\infty} \phi(\xi,t)\, dM(\xi)\;. \tag{12}$$

Then $y = d_M u$, with transfer function $D_M(s) = \int_0^\infty \frac{dM(\xi)}{s+\xi}$, for $\Re e(s) > 0$. It is a well-posed linear system, see [21].

The following energy balance can be proved:

$$\int_0^T u(t)\, y(t)\, dt = \frac{1}{2}\int_0^{+\infty} \phi(\xi,T)^2\, dM + \int_0^T\int_0^{+\infty} \xi\, \phi(\xi,t)^2\, dM\, dt\;. \tag{13}$$

The right hand side of (13) is split into two terms, a storage function evaluated at time T only, $E_\phi(T) = \frac{1}{2}\,\|\phi(T)\|^2_{H_M}$, and a dissipated energy on the time interval $(0,T)$.

A useful example can be developped with $M_\beta(d\xi) = \frac{\sin\beta\pi}{\pi}\,\xi^{-\beta}\, d\xi$ for some $0 < \beta < 1$, which provides a *diagonal realization* for the *fractional integral operator* of order β, for which $D_{M_\beta}(s) = s^{-\beta}$, to be used in the sequel with $\beta = \frac{1}{2}$.

3.1.2 Extended diffusive realizations

Let N a positive measure on $\mathbb{R}^+$ satisfying $c_N = \int_0^\infty \frac{dN}{1+\xi} < +\infty$. Consider now the dynamical system with input $u \in H^1(0,T)$, output $z \in L^2(0,T)$ and state $\widetilde{\phi} \in \widetilde{H}_N = L^2(\mathbb{R}^+, \xi\, dN)$:

$$\partial_t \widetilde{\phi}(\xi,t) \;=\; -\xi\, \widetilde{\phi}(\xi,t) \,+\, u(t)\,; \quad \widetilde{\phi}(\xi,0) \,=\, 0 \quad \forall\, \xi \in \mathbb{R}^+ \;, \tag{14}$$

$$z(t) \;=\; \int_0^{+\infty} \partial_t \widetilde{\phi}(\xi,t)\, dN(\xi) = \int_0^{+\infty} \Big[u(t) \,-\, \xi\, \widetilde{\phi}(\xi,t)\Big]\; dN(\xi)\;. \tag{15}$$

Then $z = \widetilde{d}_N u$, with transfer function $\widetilde{D}_N(s) = s\, \int_0^\infty \frac{dN(\xi)}{s+\xi}$, for $\Re e(s) > 0$.

The following energy balance can be proved:

$$\int_0^T u(t)\, z(t)\, dt = \frac{1}{2}\int_0^{+\infty} \xi\, \widetilde{\phi}(\xi,T)^2\, dN + \int_0^T\int_0^{+\infty} (u - \xi\, \widetilde{\phi})^2\, dN\, dt\;. \tag{16}$$

Again, the right hand side of (16) is split into two terms, a storage function evaluated at time T only, $\widetilde{E}_{\widetilde{\phi}}(T) = \frac{1}{2}\,\|\widetilde{\phi}(T)\|^2_{\widetilde{H}_N}$, and a dissipated energy on the time interval $(0,T)$.

Analogously, a useful example can be developped with $N_\alpha(d\xi) = M_{1-\alpha}(d\xi) = \frac{\sin\alpha\pi}{\pi}\,\xi^{-\alpha}\, d\xi$ for some $0 < \alpha < 1$, which provides a *diagonal realization* for the *fractional derivative operator* of order α, for which $D_{N_\alpha}(s) = s^{+\alpha}$, to be used in the sequel with $\alpha = \frac{1}{2}$.

3.2 An extended energy inequality

Thus, the global system (8)–(10) can be put in the abstract form $\partial_t X + \mathcal{A}\, X = 0$, where:

$$\mathcal{A}\begin{pmatrix} p \\ v \\ \varphi \\ \widetilde{\varphi} \end{pmatrix} = \begin{pmatrix} r^{-2}\,\partial_z v + \varepsilon\, \int_0^{+\infty} \varphi\, dM + \eta\, \int_0^{+\infty} [p \,-\, \xi\, \widetilde{\varphi}]\; dN \\ r^2\,\partial_z p \\ \xi\varphi - p \\ \xi\widetilde{\varphi} - p \end{pmatrix}\;; \tag{17}$$

together with the boundary conditions $p(z=0)=0$ and $v(z=1)=0$.

In order to analyze the well-posedness of this system, we introduce the state space $\mathcal{H} = L^2_p \times L^2_v \times L^2(0,1; H_M; \varepsilon\, r^2\, dz) \times L^2(0,1; \widetilde{H}_N; \eta\, r^2\, dz)$ and, with $\mathcal{V} = H^1_p \times H^1_v \times L^2(0,1; V_M; \varepsilon\, r^2\, dz) \times L^2(0,1; \widetilde{H}_N; \eta\, r^2\, dz)$, the domain of operator $\mathcal{A}$ as:

$$D(\mathcal{A}) = \left\{ (p, v, \varphi, \widetilde{\varphi})^T \in \mathcal{V}\ ,\ \left| \begin{array}{l} p(0) = 0 \\ v(1) = 0 \\ (p - \xi\varphi) \in L^2(0,1; H_M; \varepsilon\ r^2\, dz) \\ (p - \xi\widetilde{\varphi}) \in L^2(0,1; V_N; \eta\, r^2\, dz) \end{array} \right. \right\} . \qquad (18)$$

The *monotonicity* of $\mathcal{A}$ follows from the identity: $\forall X \in D(\mathcal{A})$,

$$(\mathcal{A}X,\ X)_{\mathcal{H}} \ = \ \int_0^1 \|\varphi\|^2_{\widetilde{H}_M} \varepsilon\, r^2\, dz + \int_0^1 \|p - \xi\, \widetilde{\varphi}\|^2_{H_N} \eta\, r^2\, dz \geq 0\ . \qquad (19)$$

The details of the proof of *maximality* of $\mathcal{A}$ can be found in [10]. Hence, $\mathcal{A}$ generates a $\mathcal{C}^0$-semigroup of contractions on $\mathcal{H}$.

At this stage, it is of utmost importance to notice that the resolvent of $\mathcal{A}$ is *not* compact, and a major difficulty arises in the use of LaSalle's invariance principle.

3.3 Asymptotic stability: a difficult question

Note that the asymptotic stability of the equilibrium is still an open question, in the *non*-linear case, because the precompactness of the trajectories in the energy space cannot be proved. More precisely, a *sufficient* compactness criterion, such as [14, Theorem 3.65], does not apply in our cases, because of the *unboundedness* of the ξ-domain for the diffusive equation. This is the reason why LaSalle's invariance principle (see e.g. [14, Theorem 3.64]) cannot be applied without a deeper analysis. The Hartman-Grobman-like theorem of stability in the vicinity of an hyperbolic equilibrium point is hard to apply in infinite dimension, as proposed in [3] on special cases.

In the *linear* case though, a refined analysis of the spectrum of generator of the semigroup can be performed, which allows for the use of the stability results of [1, Stability theorem], [15] or [14, Theorem 3.26]: a direct application of this result on the pseudo-differentially damped linearized pendulum, $\ddot{\vartheta} + \eta\, \partial_t^\alpha \dot{\vartheta} + \varepsilon\, \partial_t^{-\beta} \dot{\vartheta} + \omega^2\, \vartheta = 0$, can be found in [20]. Note that in [18], stability conclusions have been drawn on the Webster-Lokshin model *with constant coefficients*, using first a modal decomposition on a Riesz basis, and second, the asymptotics of the eigenfunctions of the ∂_t^α operator (of the Mittag-Leffler family), first proved in [16].

3.4 Analysis of the spectrum of the generator

In order so be self-contained, let us first recall:

Theorem 3.1. [1, Stability theorem] *Let us consider the infinitesimal generator $\mathcal{A}$ of a bounded $\mathcal{C}^0$-semigroup on a reflexive Banach space. Assume that no eigenvalue of $\mathcal{A}$ lies on the imaginary axis. If $\sigma(\mathcal{A}) \cap i\mathbb{R}$ is countable, then the semigroup generated by $\mathcal{A}$ is asymptotically stable, which means that the solutions of the differential equation $x'(t) = \mathcal{A}x(t)$ tend to 0 with $t \to \infty$.*

Note that, following our notations, the infinitesimal generator of the semigroup, defined by (17), is denoted by $-\mathcal{A}$.

3.4.1 No spectrum in the right-half plane

Following e.g. [5], since $-\mathcal{A}$ generates a contraction semigroup, it is necessarily a bounded semigroup, hence $\sigma(-\mathcal{A}) \cap \{s \in \mathbb{C},\, \Re e(s) > 0\} = \emptyset$.

3.4.2 Spectrum on the imaginary axis

As in [20], it will be necessary to make a distinction between $\lambda = 0$ and $\lambda = i\omega \neq 0$.
Step 1. Solving for $\mathcal{A}\,X = 0$ leads to $X = 0$, thanks to the boundary conditions in (18). Hence, $\lambda = 0$ is not an eigenvalue of $-\mathcal{A}$; but of course, it can be proved that $\lambda = 0$ belongs to the continuous spectrum $\sigma_c(-\mathcal{A})$.
Step 2. Now, in order to prove $\sigma(-\mathcal{A}) \cap \{i\omega,\ \omega \in \mathbb{R},\ \omega \neq 0\} = \emptyset$, we show the continuity of the resolvent $(i\omega\, I + \mathcal{A})^{-1}$ on $\mathcal{H}$. Following the techniques developped in [10, chap. 2], given any $Y = (f, g, \chi, \widetilde{\chi}) \in \mathcal{H}$, we seek some $X = (p, v, \phi, \widetilde{\phi}) \in D(\mathcal{A})$, such that $(i\omega\, I + \mathcal{A})\, X = Y$. For the last two components ϕ and $\widetilde{\phi}$, an algebraïc inversion if perfomed, which gives rise to a set of two coupled first-order equations in the unknowns (p, v). In order to solve them, we first replace $v = (i\omega)^{-1}(g - r^2\, \partial_z p)$ and get a weak version of these two in the following *variational form*:

$$\begin{cases} \text{Find } p \in H^1_p \text{ such that:} \\ a_\omega(p, q) = l_\omega(q), \quad \forall q \in H^1_p, \end{cases} \tag{20}$$

where

$$\begin{aligned} a_\omega(p,q) &:= \int_0^1 [\Omega(z) p\, \bar{q} + \partial_z p\, \partial_z \bar{q}]\; r^2\, dz\,, \quad l_\omega(q) := i\omega \int_0^1 h_\omega\, \bar{q}\, r^2\, dz + \int_0^1 g\, \partial_z \bar{q}\, dz; \\ \Omega(z) &:= -\omega^2 + i\omega\, \varepsilon(z) \int_0^\infty \frac{1}{i\omega + \xi}\, dM_\beta - \omega^2\, \eta(z) \int_0^\infty \frac{1}{i\omega + \xi}\, dM_{1-\alpha}; \\ h_\omega(z) &:= f(z) - \varepsilon(z) \int_0^\infty \frac{1}{i\omega + \xi}\, \chi(z,\xi)\, dM_\beta + \eta(z) \int_0^\infty \frac{1}{i\omega + \xi}\, \widetilde{\chi}(z,\xi)\, \xi\, dM_{1-\alpha}. \end{aligned}$$

First, a_ω is a continuous sesquilinear form on $H^1_p \times H^1_p$; second, it can easily be proved that $h \in L^2_p$, thus l_ω is a continuous antilinear form on H^1_p; but, unfortunately $\Re e(\Omega(z)) = -\omega^2 + \omega^2\, \varepsilon(z) \int_0^\infty \frac{1}{\omega^2+\xi^2}\, dM_\beta - \omega^2\, \eta(z) \int_0^\infty \frac{\xi}{\omega^2+\xi^2}\, dM_{1-\alpha}$ has no definite sign, thus we can *not* apply the complex version of Lax-Milgram theorem in order to conclude.
Therefore, we have to resort to the *Fredholm alternative*: indeed, (20) can be rewritten as $-(\mathcal{K}p, q)_{H^1_p} + (p, q)_{H^1_p} = l_\omega(q)$, or equivalently $-\mathcal{K}p + p = L_\omega$, where $\mathcal{K}$ is a *compact* operator in H^1_p, and $L_\omega \in H^1_p$ is given by Riesz representation theorem for the continuous antilinear form l_ω. It remains to show that 1 is not an eigenvalue of $\mathcal{K}$, which turns out to be true, since, inspecting the imaginary part of the problem, we get

$$\omega \int_0^1 \left[\varepsilon(z) \int_0^\infty \frac{\xi}{\omega^2 + \xi^2}\, dM_\beta + \omega^2\, \eta(z) \int_0^\infty \frac{1}{\omega^2 + \xi^2}\, dM_{1-\alpha}\right] |p(z)|^2\, r^2\, dz = 0,$$

which implies $p = 0$ in H^1_p, since both ε and η are strictly positive.
Conclusion. From the two previous steps, and using [1, Stability theorem], we can conclude to the *asymptotic stability* of the Webster-Lokshin model (8)-(10) in $\mathcal{H}$.

4 CONCLUSIONS AND FUTURE WORKS

4.1 Conclusions

The first aim of this paper is to show that systems theory, namely realization theory (either finite- or infinite-dimensional), cannot be bypassed in the analysis of complex systems. Conversely, the second aim is to show that drawing conclusions on the stability of an infinite-dimensional dynamical system from a formal energy analysis can only be misleading (if not false), and that one must resort to more sophisticated theorems. This shows the necessary interplay between mathematical analysis and systems theory.

More precisely, in this paper we have been able to prove the asymptotic internal stability of a linear Webster-Lokshin model, which uses some infinite-dimensional realization theory on the one hand, and requires to resort to Arendt-Batty stability theorem on the other hand.

4.2 Future Works

A first useful generalization of this result is the combination of an impedance as in § 2.2, with some fractional damping as in § 3: the well-posedness of the global system has already been proved in [8, 10], hence only the spectral analysis of the generator of the extended semigroup needs to be performed, this should not be a major difficulty.

A whole family of PDE models with fractional or diffusive damping has been introduced in [17], their well-posedness was proved, but the asymptotic stability still needed to be proved properly: using Arendt-Batty theorem should definitely help in this respect.

Another important question to be investigated is the following. In Section § 2.2.1, by assuming that the impedance $\mathcal{Z}_i$ are of rational type, and by using the Kalman-Yakubovich-Popov lemma, we get the existence of a minimal realization of finite dimension. A natural question is the following: for more general positive real impedances, which are irrational functions, does their exists a realization in infinite dimension (see e.g. [25]), minimal in some sense (see e.g. [2]) ? If yes, how can it help in order to achieve our stability analysis?

Now for non-linear systems, such as the Burgers-Webster-Lokshin equation which accounts for the propagation of pressure waves in a brass wind-instrument, the well-posedness could be proved using e.g. [9], but the stability question remains open, even though (non-quadratic) energy techniques seem to provide the desired result, up to some missing compactness argument!

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