On the Construction of Positive Quadratic Forms using Semidefinite Programming

Matthew M. Peet

EDP-MOSAR

March 22, 2007
INRIA - Rocquencourt
Projet SOSSO2

Consider: A System of Linear Differential Equations with Discrete Delays

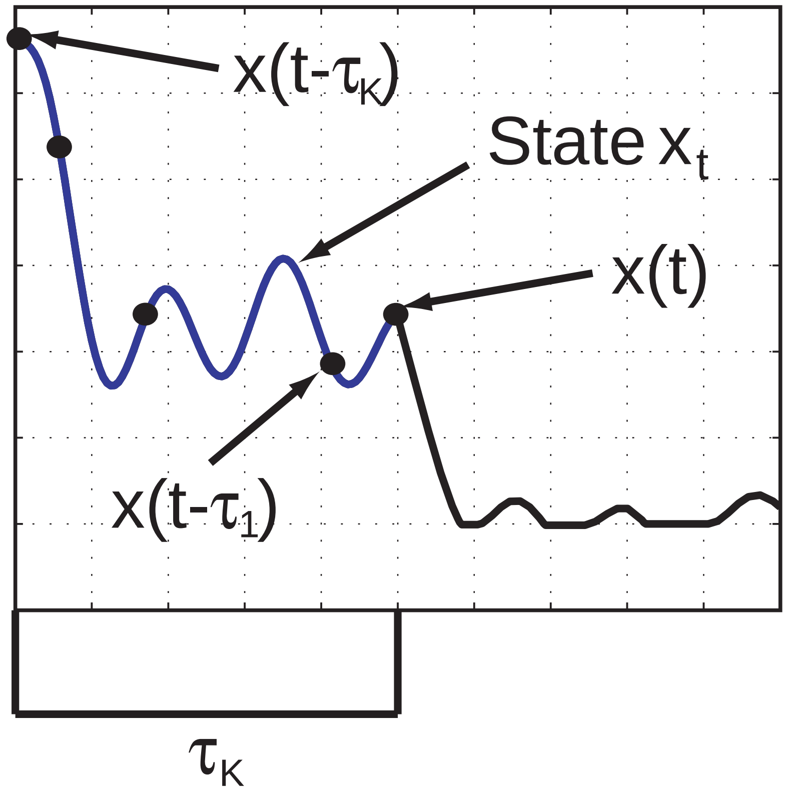
$$\dot{x}(t) = \sum_{i=0}^{m} A_i x(t - \tau_i)$$

Problem: Stability

Given $A_i \in \mathbb{R}^{n \times n}$, $\tau_i \in \mathbb{R}^+$,
does $\lim_{t \to \infty} x(t) = 0$
for any x_0?

State Space: Systems with Discrete Delays

$$\dot{x}(t) = g\big(x(t), x(t-\tau_1), \ldots, x(t-\tau_K)\big)$$



Lyapunov Functions: A Method of Stability Analysis

Lyapunov functions are used to prove stability of functional differential equations.

Theorem 1 *A functional differential system is stable if there exists* $V : \mathcal{C}_\tau \to \mathbb{R}$ *and* $\epsilon > 0$ *such that for all* $x_t \in \mathcal{C}_\tau$*, we have*

$$V(x_t) \geq \epsilon \|x(t)\|_2$$
$$\dot{V}(x_t) \leq 0.$$

$\dot{V}$ is the Lie derivative.

x_t is the state at time t.

The Approach:

The set of positive V is convex.
The set of negative $\dot{V}$ is convex.
$\Rightarrow$ If the map $V \mapsto \dot{V}$ is linear, then we have convex optimization... more to come.

Stability of Linear Differential Equations with Delay:

$$\dot{x}(t) = \sum_{i=0}^{m} A_i x(t - \tau_i)$$

Stable iff $\exists\, V > 0 : \dot{V} < 0$, where

$$V(x) = \int_{-\tau_m}^{0} \begin{bmatrix} x(0) \\ x(s) \end{bmatrix}^T M(s) \begin{bmatrix} x(0) \\ x(s) \end{bmatrix} ds$$

$$+ \int_{-\tau_m}^{0} \int_{-\tau_m}^{0} x(s) N(s,t) x(t) ds dt$$

Problem: Find M and N so that for all $x \in \mathcal{C}$:

$$V(x) > 0$$

$$\dot{V}(x) < 0$$

Its Convex, but hard....

Computational Complexity: Is it NP-Hard?

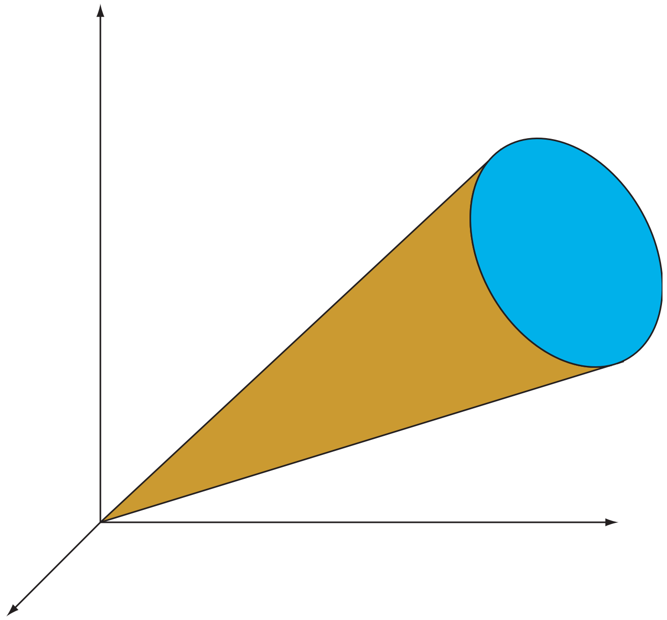
Problems in P:

- The shortest path
- Stability of linear systems in finite dimensions
- Linear Programming
- Semidefinite programming?

Problems in NP+:

- The traveling salesman
- Matrix Copositivity
- Positivity of Polynomials
- μ
- Delay-Independent Stability

Convex Optimization



Problem:

$$\max \; bx$$
$$\text{subject to} \quad Ax \in C$$

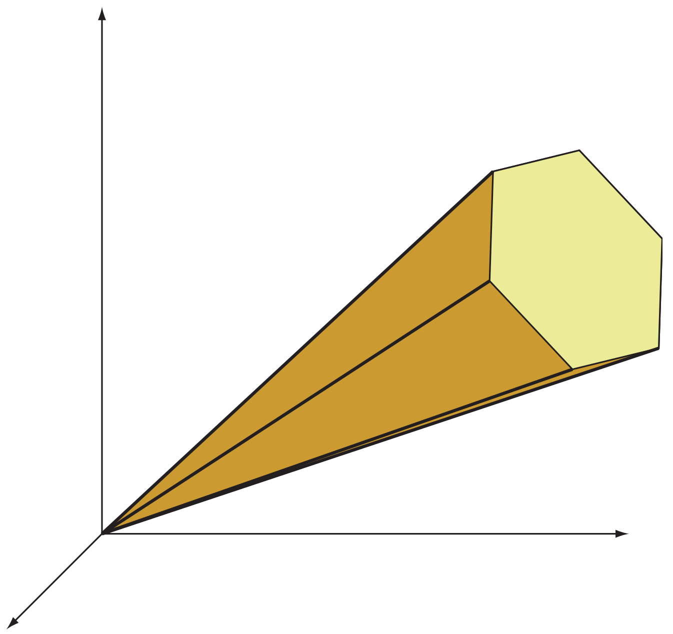
The problem is *convex optimization* if

- C is a convex cone.
- b and A are affine.

Complexity: Convex Optimization over C is, in general, tractable if

- There is an efficient set membership test for $x \in C$

Semidefinite Programming(SDP)



Problem:

$$\max\ b^T x$$

subject to $\quad A_0 + \sum_{i=1}^{m} A_i x_i \succeq 0$

Here

- $x \in \mathbb{R}^m$ and the A_i are symmetric matrices.
- $\succeq 0$ denotes the cone of positive semidefinite matrices.

Computationally Tractable

Semidefinite Programming(SDP): Common Examples in Control

- Stability

$$A^T X + XP \prec 0$$
$$X \succ 0$$

- Stabilization

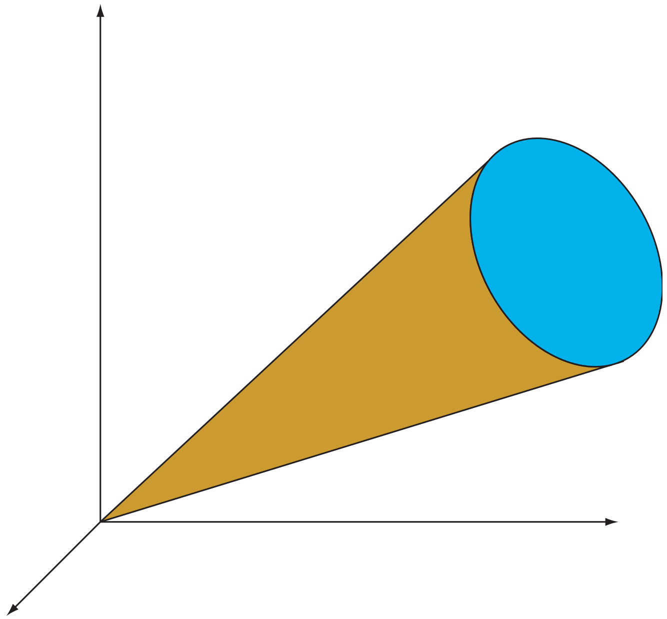
$$AX + BZ + XA^T + Z^T B^T \prec 0$$
$$X \succ 0$$

- H_2 Synthesis

$$\min Tr(W)$$
$$\begin{bmatrix} A & B_2 \end{bmatrix} \begin{bmatrix} X \\ Z \end{bmatrix} + \begin{bmatrix} X & Z^T \end{bmatrix} \begin{bmatrix} A^T \\ B_2^T \end{bmatrix} + B_1 B_1^T \prec 0$$
$$\begin{bmatrix} X & (CX + DZ)^T \\ (CX + DZ) & W \end{bmatrix} \succ 0$$

- KYP Lemma

Polynomial Programming



Problem:

$$\max \; c^T x$$

subject to $A_0(y) + \sum_i^n x_i A_i(y) \succeq 0 \quad \forall y$

The A_i are matrices of polynomials in y. i.e. Using multi-index notation,

$$A_i(y) = \sum_\alpha A_{i,\alpha}\, y^\alpha$$

Computationally Intractable

Polynomial Programming: Examples

- Stability of Nonlinear Systems

$$f(y)^T \nabla p(y) < 0$$
$$p(y) > 0$$

- Matrix Copositivity

$$y^T M y - g(y)^T y \geq 0$$
$$g(y) \geq 0$$

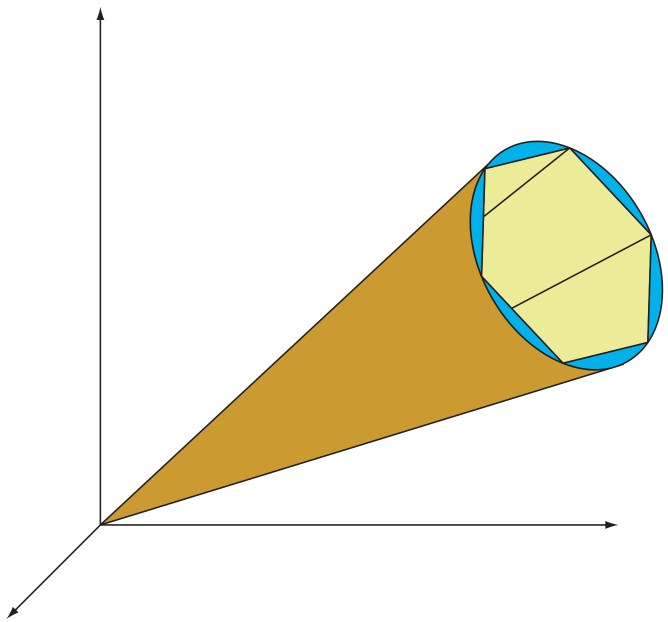
- Integer Programming

$$\max \gamma$$
$$p_0(y)(\gamma - f(y)) - (\gamma - f(y))^2 + \sum_{i=1}^n p_i(y)(y_i^2 - 1) \geq 0$$
$$p_0(y) \geq 0$$

- Also μ

Positivstellensatz results are commonly used to set up these problems.

Sum-of-Squares(SOS) Programming



Problem:

$$\max\ c^T x$$

subject to $\quad A_0(y) + \sum_i^n x_i A_i(y) \in \Sigma_s$

- Σ_s is the cone of *sum-of-squares* matrices. If $S \in \Sigma_s$, then for some $G_i \in \mathbb{R}[x]$,

$$S(y) = \sum_{i=1}^{r} G_i(y)^T G_i(y)$$

Computationally Tractable: $S \in \Sigma_s$ is an SDP constraint.

SOS Programming: Why is $M \in \Sigma_s$ an SDP?

Define $Z_d(x)$ to be the vector of monomial bases in dimension n of degree d or less.

For example, if $n = 1$, and $x \in \mathbb{R}^2$, then

$$Z_2(x)^T = \begin{bmatrix} 1 & x_1 & x_2 & x_1x_2 & x_1^2 & x_2^2 \end{bmatrix}$$

If $n = 2$, and $x \in \mathbb{R}^2$, then

$$Z_1(x)^T = \begin{bmatrix} 1 & x_1 & x_2 & & & \\ & & & 1 & x_1 & x_2 \end{bmatrix}$$

Lemma 1 *Suppose M is polynomial of degree $2d$. $M \in \Sigma_s$ iff there exists some $Q \succeq 0$ such that*

$$M(x) = Z_d(x)^T Q Z_d(x).$$

Note: Sometimes we won't mention d explicitly.

SOS Programming: Example

$$M(y,z) = \begin{bmatrix} (y^2+1)z^2 & yz \\ yz & y^4+y^2-2y+1 \end{bmatrix}$$

Problem: Is $M \in \Sigma_s$?

Algorithm:

Step 1: Write

$$M(y,z) = NZ_4(y,z)$$

Step 2: Construct B such that if $N = B\,\mathrm{vec}(Q)$, then

$$NZ_4(y,z) = Z_2(y,z)^T Q Z_2(y,z)$$

This only depends on Z_2 and Z_4

Step 3: Find $Q \succ 0$ such that $N = B\,\mathrm{vec}(Q)$

SOS Programming: Solution

$$M(y,z)=\begin{bmatrix}(y^2+1)z^2 & yz\\ yz & y^4+y^2-2y+1\end{bmatrix}$$

$$\begin{bmatrix}(y^2+1)z^2 & yz\\ yz & y^4+y^2-2y+1\end{bmatrix}=\begin{bmatrix}z&0\\yz&0\\0&1\\0&y\\0&y^2\end{bmatrix}^T\begin{bmatrix}1&0&0&0&1\\0&1&1&-1&0\\0&1&1&-1&0\\0&-1&-1&1&0\\1&0&0&0&1\end{bmatrix}\begin{bmatrix}z&0\\yz&0\\0&1\\0&y\\0&y^2\end{bmatrix}$$

$$=\begin{bmatrix}z&0\\yz&0\\0&1\\0&y\\0&y^2\end{bmatrix}^T\begin{bmatrix}0&1&1&-1&0\\1&0&0&0&1\end{bmatrix}^T\begin{bmatrix}0&1&1&-1&0\\1&0&0&0&1\end{bmatrix}\begin{bmatrix}z&0\\yz&0\\0&1\\0&y\\0&y^2\end{bmatrix}=\begin{bmatrix}yz&1-y\\z&y^2\end{bmatrix}^T\begin{bmatrix}yz&1-y\\z&y^2\end{bmatrix}\in\Sigma_s$$

The First Step: How to Construct SOS Programs?

Few questions are naturally expressed as polynomial programs.

- Instead consider optimization over semialgebraic sets

$$\begin{aligned} \max f(x) &: \\ p_i(x) &\geq 0 \\ q_i(x) &= 0 \end{aligned}$$

Special cases include:

- Matrix Copositivity:

$$\begin{aligned} \min x^T M x &: \\ x &\geq 0 \end{aligned}$$

- Integer programming:

$$\begin{aligned} \max f(x) &: \\ x_i^2 &= 1 \end{aligned}$$

The Next Step: Positivstellensatz

Theorem 2 (Stengle) *The following are equivalent*

- $$\left\{ x : \begin{matrix} p_i(x) \geq 0 & i = 1, \ldots, k \\ q_j(x) = 0 & j = 1, \ldots, m \end{matrix} \right\} = \emptyset$$

- *There exist* $t_i \in \mathbb{R}[x]$, $s_i, r_{ij}, \ldots \in \Sigma_s$ *such that*

$$-1 = \sum_i q_i t_i + s_0 + \sum_i s_i p_i + \sum_{i \neq j} r_{ij} p_i p_j + \cdots$$

The Next Step: Positivstellensatz

Let

$$\mathcal{P} := \left\{ x : \begin{array}{ll} p_i(x) \geq 0 & i = 1, \dots, k \\ q_j(x) = 0 & j = 1, \dots, m \end{array} \right\}$$

Theorem 3 (Putinar) *Suppose $\mathcal{P}$ is "compact+". Suppose $f(x) \geq 1$ for $x \in \mathcal{P}$. Then there exist $s_i \in \Sigma_s$ and $t_i \in \mathbb{R}[x]$ such that*

$$f(x) - \sum_{i=1}^{k} s_i(x) p_i(x) + \sum_{i=1}^{m} t_i(x) q_i(x) \in \Sigma_s$$

There are many other formulations

Example: Robust Lyapunov Stability

Problem: Is $\dot{x}(t) = f(\alpha, x(t))$ stable for $\alpha \in \Delta := \{\alpha : \|\alpha\|^2 \leq 1\}$?

$$\text{find } V: \quad \text{for any } \alpha \in \Delta,$$
$$V(x,\alpha) \geq \epsilon\|x\|^2$$
$$\dot{V}(x,\alpha) \leq 0$$

Equivalently:

$$\text{find } V(x,\alpha):$$
$$f(\alpha,x)^T \nabla V(x,\alpha) \leq 0 \quad \text{for } \alpha \in \Delta$$
$$V(x,\alpha) \geq \epsilon\|x\|^2$$

SOS Program:

$$\text{find } V:$$
$$-f(\alpha,x)^T \nabla_x V(x,\alpha) - s(\alpha,x)(\|\alpha\|^2 - 1) \in \Sigma_s$$
$$V(x,\alpha) - \epsilon\|x\|^2 \in \Sigma_s, \qquad s(\alpha,x) \in \Sigma_s$$

Return to Linear Differential Equations with Delay:

$$\dot{x}(t) = \sum_{i=0}^{m} A_i x(t - \tau_i)$$

Stable iff $\exists\, V > 0 : \dot{V} < 0$, where

$$V(x) = \int_{-\tau_m}^{0} \begin{bmatrix} x(0) \\ x(s) \end{bmatrix}^T M(s) \begin{bmatrix} x(0) \\ x(s) \end{bmatrix} ds + \int_{-\tau_m}^{0} \int_{-\tau_m}^{0} x(s) N(s,t) x(t) ds dt$$

Problem: Find M and N so that:

$$V(x) > 0$$
$$\dot{V}(x) < 0$$

Result: Positivity of Part 1

Theorem 4 *Let M be piecewise-continuous, then following are equivalent*

1. *There exists some $\epsilon > 0$ so that*
$$\int_{-\tau_m}^{0} \begin{bmatrix} x(0) \\ x(s) \end{bmatrix}^T M(s) \begin{bmatrix} x(0) \\ x(s) \end{bmatrix} ds \geq \epsilon \|x\|^2$$
2. *There exists a function T and $\epsilon' > 0$ such that*
$$\int_{-\tau_m}^{0} T(s)ds = 0 \quad \textit{and} \quad M(s) + \begin{bmatrix} T(s) & 0 \\ 0 & 0 \end{bmatrix} \succeq \epsilon' I$$

Computationally Tractable:

- Assume M and T are polynomials
- The constraint $\int_{-\tau_m}^{0} T(s)ds = 0$ is linear
- For the 1-D case, Σ_s is exact.

Example: Positive Multipliers

$$M(s) = \begin{bmatrix} -2s^2+2 & s^3-s \\ s^3-s & s^4+s^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ s & 0 \\ 0 & s \\ 0 & s^2 \end{bmatrix}^T \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ s & 0 \\ 0 & s \\ 0 & s^2 \end{bmatrix} + \begin{bmatrix} -1 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ s & 0 \\ s^2 & 0 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ s & 0 \\ 0 & s \\ 0 & s^2 \end{bmatrix}^T \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{bmatrix}^T \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ s & 0 \\ 0 & s \\ 0 & s^2 \end{bmatrix} + \begin{bmatrix} 3s^2-1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} s & s^2 \\ 1 & -s \end{bmatrix}^T \begin{bmatrix} s & s^2 \\ 1 & -s \end{bmatrix} + \begin{bmatrix} 3s^2-1 & 0 \\ 0 & 0 \end{bmatrix}$$

And

$$\begin{bmatrix} -1 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{1}{2} & 0 \\ \frac{1}{3} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad \Rightarrow \qquad \int_{-1}^{0} (3s^2-1)ds = 0$$

Result: Positivity of Part 2

Theorem 5 *Suppose $N(s,t)$ is a polynomial. Then the following are equivalent:*

- $$\int_{-\tau_m}^{0}\int_{-\tau_m}^{0} x(s)^T N(s,t) x(t) ds dt \geq 0 \qquad \textit{for all } x \in \mathcal{C}$$
- *There exists a $Q \geq 0$ such that*
$$N(s,t) + N(t,s)^T = Z(s)^T Q Z(t)$$

Notes:

- Map is affine
- N is separable

Example: Positive Integral Operators

If

$$N(s,t)=\begin{bmatrix}1-t-s+2st & 1-s-st^2\\ 1-t-s^2t & 1+s^2t^2\end{bmatrix}=\begin{bmatrix}1&0\\s&0\\0&1\\0&s^2\end{bmatrix}^T\begin{bmatrix}1&-1&1&0\\-1&2&-1&-1\\1&-1&1&0\\0&-1&0&1\end{bmatrix}\begin{bmatrix}1&0\\t&0\\0&1\\0&t^2\end{bmatrix}$$

$$=\begin{bmatrix}1&0\\s&0\\0&1\\0&s^2\end{bmatrix}^T\begin{bmatrix}1&-1&1&0\\0&-1&0&1\end{bmatrix}^T\begin{bmatrix}1&-1&1&0\\0&-1&0&1\end{bmatrix}\begin{bmatrix}1&0\\t&0\\0&1\\0&t^2\end{bmatrix}=\begin{bmatrix}1-s&1\\-s&s^2\end{bmatrix}^T\begin{bmatrix}1-t&1\\-t&t^2\end{bmatrix}$$

Then

$$\int_{-\tau}^{0}\int_{-\tau}^{0}x(s)^TN(s,t)x(t)dsdt=\int_{-\tau}^{0}\int_{-\tau}^{0}x(s)^TG(s)^TG(t)x(t)dsdt$$

$$=\int_{-\tau}^{0}x(s)^TG(s)^Tds\int_{-\tau}^{0}G(t)x(t)dt=K^TK\geq 0$$

Result: Further Improvements

- **Joint Positivity:**

  Does $V \geq 0$ imply that both M and N define positive functionals?

- **Semi-Separable Kernels:**

  Suppose

$$N(t,s) = \begin{cases} N_1(t,s) & s < t \\ N_2(t,s) & t \leq s \end{cases}$$

Example: Standard Test Case - Multiple Delays

We now consider a system with multiple delays.

$$\dot{x}(t) = \begin{bmatrix} -2 & 0 \\ 0 & -\frac{9}{10} \end{bmatrix} x(t) + \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix} \left[\frac{1}{20} x(t - \frac{\tau}{2}) + \frac{19}{20} x(t - \tau) \right]$$

A bisection method was used and results are listed below.

Our Approach			Piecewise Functional		
d	$\tau_{\min}$	$\tau_{\max}$	N_2	$\tau_{\min}$	$\tau_{\max}$
1	.20247	1.354	1	.204	1.35
2	.20247	1.3722	2	.203	1.372
Analytic	.20246	1.3723			

Table 1: τ_{max} and τ_{min} using a piecewise-linear functional and our approach and compared to the analytical limit.

Generalization: Uncertainty

Parameter-Dependent Lyapunov: Replace $M(s)$ and $N(s,t)$ with $M(s,\alpha)$ and $N(s,t,\alpha)$.

Use the Positivstellensatz: Applied to the constraints

$$M(s,\alpha)+T(s,\alpha)\succeq 0 \qquad \text{for all } \alpha\in\Delta$$

and

$$\begin{aligned} N(s,t,\alpha) &= Z(s)^T Q(\alpha) Z(t) \\ Q(\alpha) &\succeq 0 \qquad\qquad \text{for all } \alpha\in\Delta. \end{aligned}$$

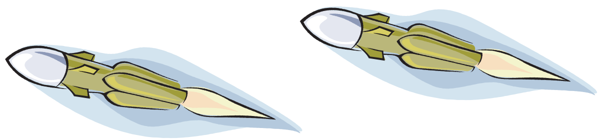
Example: Standard Test Case - Uncertain Delay

By including τ as an uncertain parameter in the Lyapunov functionals, we can prove stability over the interval $[\tau_{\min}, \tau_{\max}]$ directly.

d in τ	d in θ	$\tau_{\min}$	$\tau_{\max}$
1	1	.1002	1.6246
1	2	.1002	1.717
Analytic		.10017	1.71785

Table 2: Stability on the interval $[\tau_{\min}, \tau_{\max}]$ vs. degree using a parameter-dependent functional

Example: Remote Control



A Simple Inertial System: Suppose we are given a specific type of PD controller that we want to implement.

$$\ddot{x}(t) = -ax(t) - \frac{a}{2}\dot{x}(t)$$

The controller is stable for all positive a. Now suppose we want to maintain control from a remote location. When we include the **communication delay**, the equation becomes.

$$\ddot{x}(t) = -ax(t-\tau) - \frac{a}{2}\dot{x}(t-\tau)$$

Question: For what range of a and τ will the controller be stable. The model is linear, but contains a parameter and an uncertain delay.

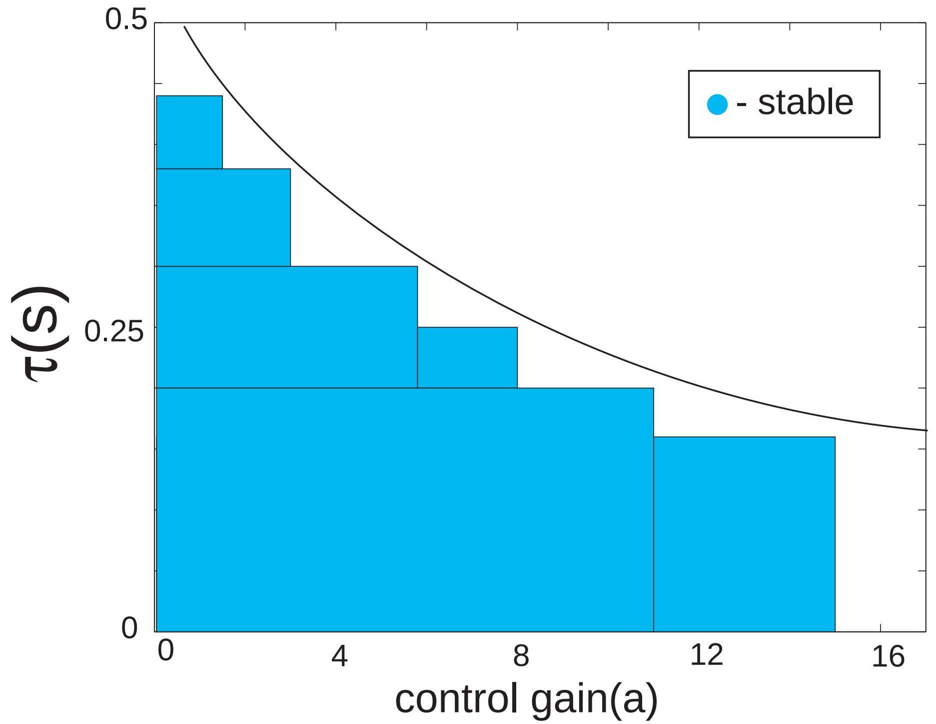
Example: Remote Control

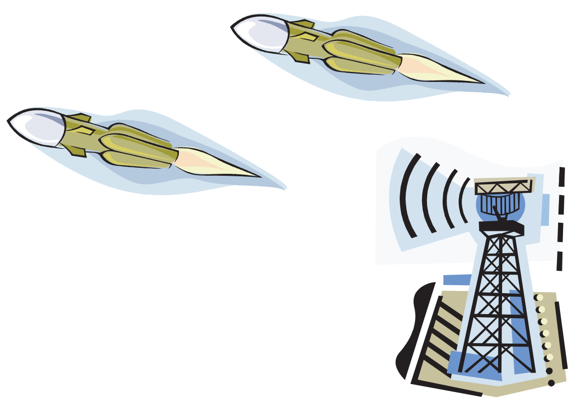
Recall that we considered an inertial system controlled remotely using PD control

$$\ddot{x}(t) = -ax(t-\tau) - \frac{a}{2}\dot{x}(t-\tau)$$

Question: For what range of a and τ will the controller be stable?

- We use parameter-dependent functionals.





Generalization: Nonlinear Time-Delay Systems

Consider nonlinear systems which have discrete delays.

$$\dot{x}(t) = f(x(t), x(t-\tau_1), \ldots, x(t-\tau_K))$$

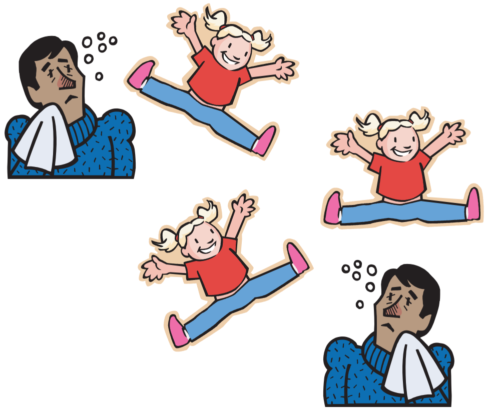
Here we assume $x(t) \in \mathbb{R}^n$ and f is polynomial.

Generalize the quadratic functional:

$$V(\phi) := \int_{-\tau_K}^{0} Z(\phi(0),\phi(\theta))^T M(\theta) Z(\phi(0),\phi(\theta)) d\theta \\ + \int_{-\tau_K}^{0}\int_{-\tau_K}^{0} Z(\phi(\theta))^T R(\theta,\omega) Z(\phi(\omega)) d\theta d\omega$$

Computation: The same constraints on M and N are sufficient for positivity.

Example: Epidemiological Model of Infection



Consider a human population subject to non-lethal infection by a virus. The disease has **incubation period** (τ). Cooke(1978) models the percentage of infected humans(y) using the following equation.

$$\dot{y}(t) = -ay(t) + by(t-\tau)\left[1 - y(t)\right]$$

Where

- a is the rate of recovery for infected persons
- b is the rate of infection for exposed people

The model is nonlinear and contains delay. Equilibria exist at $y^* = 0$ and $y^* = (b-a)/b$.

Example: Epidemiological Model

Recall the dynamics of infection are given by

$$\dot{y}(t) = -ay(t) + by(t-\tau)\left[1 - y(t)\right]$$

Cooke used the following Lyapunov functional to prove delay-independent stability of the 0 equilibrium for $a > b > 0$.

$$V(\phi) = \frac{1}{2}\phi(0)^2 + \frac{1}{2}\int_{-\tau}^{0} a\phi(\theta)^2 d\theta$$

Our method was able to prove delay-independent stability for $a > b > 0$ using the following functional.

$$V(\phi) = 1.75\phi(0)^2 + \int_{-\tau}^{0} (1.47a + .28b)\phi(\theta)^2 d\theta$$

Interpretation: When the rate of recovery is greater than the rate of infection, the epidemic will die out.

Research Directions

Theory

- Stabilizing Controllers
- Partial Differential Equations
- Optimal Controller Synthesis
- The KYP lemma

Applications

Industrial and Electrical:

- Communication Systems
- Manufacturing

Biological:

- Cancer Therapy
- HIV Therapy