Stabilization of infinite-dimensional linear systems: A fractional ideal approach

Alban Quadrat

INRIA Sophia Antipolis, APICS project,

2004 route des lucioles, BP 93, 06902 Sophia Antipolis cedex, France.

Alban.Quadrat@sophia.inria.fr

www-sop.inria.fr/cafe/Alban.Quadrat/index.html

Groupe de Travail EDP – GDR MACS,
Lyon (17-18/01/08)

Transfer functions

- Ordinary differential equation:

$$\dot{z}(t) = z(t) + u(t), \ \ z(0) = 0 \quad \Rightarrow \ \widehat{z}(s) = \frac{1}{(s-1)}\, \widehat{u}(s).$$

- Differential time-delay equation:

$$\left\{ \begin{array}{l} \dot{z}(t) = z(t) + u(t), \ \ x(0) = 0, \\ y(t) = \left\{ \begin{array}{ll} 0, & 0 \leq t < h, \\ z(t-h), & t \geq h, \end{array} \right. \end{array} \right. \quad \Rightarrow \ \widehat{y}(s) = \frac{e^{-h\,s}}{(s-1)}\, \widehat{u}(s).$$

- Partial differential equation:

$$\left\{ \begin{array}{l} \dfrac{\partial^2 z}{\partial t^2}(x,t) - a^2\, \dfrac{\partial^2 z}{\partial x^2}(x,t) = 0, \\ z(x,0) = 0, \quad \dfrac{\partial z}{\partial t}(x,0) = 0, \\ z(0,t) = u(t), \quad z(l,t) = 0, \\ y(t) = z(\overline{x},t), \end{array} \right. \quad \Rightarrow \ \widehat{y}(s) = \frac{\left(e^{-\frac{\overline{x}}{a}\,s} - e^{-\frac{(2\,l-\overline{x})\,s}{a}}\right)}{\left(1 - e^{-\frac{2\,a}{l}\,s}\right)}\, \widehat{u}(s).$$

Examples of transfer functions

- Heat equation:

$$\left\{\begin{array}{l} \dfrac{\partial z}{\partial t}(x,t) - \lambda^2 \, \dfrac{\partial^2 z}{\partial x^2}(x,t) = 0, \\ z(x,0) = 0, \\ z(0,t) = u(t), \quad z(l,t) = 0, \\ y(t) = z(\overline{x}, t), \end{array}\right. \Rightarrow \; \widehat{y}(s) = \frac{\left(e^{\lambda\,(l-\overline{x})\,\sqrt{s}} - e^{-\lambda\,(l-\overline{x})\,\sqrt{s}}\right)}{\left(e^{\lambda\,l\,\sqrt{s}} - e^{-\lambda\,l\,\sqrt{s}}\right)} \, \widehat{u}(s).$$

- Telegraph equation:

$$\left\{\begin{array}{l} \dfrac{\partial^2 z}{\partial t^2}(x,t) - a^2 \, \dfrac{\partial^2 z}{\partial x^2}(x,t) - k\, z(x,t) = 0, \\ z(x,0) = 0, \quad \dfrac{\partial z}{\partial t}(x,0) = 0, \\ z(0,t) = u(t), \quad \lim_{x \to +\infty} z(x,t) = 0, \\ y(t) = z(\overline{x}, t), \end{array}\right. \Rightarrow \; \widehat{y}(s) = e^{\frac{-\sqrt{s^2 - k}}{a}\,\overline{x}} \, \widehat{u}(s).$$

Signal spaces

- Let us define the right half plane $\mathbb{C}_+ = \{s \in \mathbb{C} \mid \mathrm{Re}\, s > 0\}$.
- The Hardy algebra $H^\infty(\mathbb{C}_+)$ is defined by:

$$H^\infty(\mathbb{C}_+) = \{\text{analytic functions } f \text{ in } \mathbb{C}_+ \mid \ \| f \|_\infty = \sup_{s \in \mathbb{C}_+} |f(s)| < +\infty\}.$$

$H^\infty(\mathbb{C}_+)$ is a commutative Banach algebra.

- The Hardy $\mathbb{C}$-vector space $H^2(\mathbb{C}_+)$ is defined by:

$$H^2(\mathbb{C}_+) = \{\text{analytic functions } f \text{ in } \mathbb{C}_+ \mid$$

$$\| f \|_2 = \sup_{x \in \mathbb{R}_+} \left(\int_{-\infty}^{+\infty} |f(x+iy)|^2 dy \right)^{1/2} < +\infty\}$$

$H^2(\mathbb{C}_+)$ is a Hilbert space and $H^2(\mathbb{C}_+) = \mathcal{L}(L^2(\mathbb{R}_+))$, where:

$$L^2(\mathbb{R}_+) = \{g : \mathbb{R}_+ \to \mathbb{R} \mid \int_0^{+\infty} |g(t)|^2 \, dt < +\infty\}.$$

Signal spaces

- $L^1(\mathbb{R}_+) = \{f : [0, +\infty[\to \mathbb{R} \mid \ \| f \|_1 = \int_0^{+\infty} |f(t)| \, dt < +\infty\}$,

$$l^1(\mathbb{Z}_+) = \{a : \mathbb{Z}_+ = \{0, 1, \ldots\} \to \mathbb{R} \mid \ \| (a_i)_{i \in \mathbb{Z}_+} \|_1 = \sum_{i=0}^{+\infty} |a_i| < +\infty\}.$$

- **Definition:** The Wiener algebra $\mathcal{A}$ is defined by:

$$\mathcal{A} = \{f = g + \textstyle\sum_{i=0}^{+\infty} a_i \, \delta_{(t-h_i)} \mid \quad g \in L^1(\mathbb{R}_+), \ (a_i)_{i \in \mathbb{Z}_+} \in l^1(\mathbb{Z}_+), \\ 0 = h_0 \leq h_1 \leq h_2 \leq \ldots\}.$$

- $\mathcal{A}$ is a commutative Banach algebra w.r.t.:

$$\| f \|_{\mathcal{A}} = \| g \|_1 + \| (a_i)_{i \in \mathbb{Z}_+} \|_1 .$$

- $\widehat{\mathcal{A}} = \{\mathcal{L}(f) \mid f \in \mathcal{A}\}, \quad \| \widehat{f} \|_{\widehat{\mathcal{A}}} = \| f \|_{\mathcal{A}}.$

$L^2(\mathbb{R}_+) - L^2(\mathbb{R}_+)$-stability

- **Theorem**:

1. $\forall\, a,\, b \in H^\infty(\mathbb{C}_+),\ \forall\, f,\, g \in H^2(\mathbb{C}_+):\ a\,f + b\,g \in H^2(\mathbb{C}_+)$.
2. Let $h = \frac{n}{d}$, $0 \neq d,\, n \in H^\infty(\mathbb{C}_+)$. Then, linear operator

$$\begin{array}{rcl} \Lambda : H^2(\mathbb{C}_+) & \longrightarrow & H^2(\mathbb{C}_+), \\ u & \longmapsto & h\,u, \end{array}$$

is bounded, i.e.,

$$\mathrm{dom}(\Lambda) = \{u \in H^2(\mathbb{C}_+) \mid \Lambda(u) \in H^2(\mathbb{C}_+)\} = H^2(\mathbb{C}_+),$$

iff $h \in H^\infty(\mathbb{C}_+)$. Then, we have:

$$\|\,\Lambda\,\|_{\mathcal{L}\,(H^2(\mathbb{C}_+),\, H^2(\mathbb{C}_+))} \;=\; \sup_{0 \neq u \in H^2(\mathbb{C}_+)} \frac{\|\,h\,u\,\|_2}{\|\,u\,\|_2} = \|\,h\,\|_\infty.$$

$L^\infty(\mathbb{R}_+) - L^\infty(\mathbb{R}_+)$-stability

- **Theorem:** Let $p \in [1, +\infty]$.

1. $\forall\, a,\, b \in \mathcal{A}, \quad \forall\, f,\, g \in L^p(\mathbb{R}_+): \quad a \star f + b \star g \in L^p(\mathbb{R}_+)$.
2. Let $h = n \star d^{-1}$, $0 \neq d$, $n \in \mathcal{A}$. Then, the linear operator

$$\Lambda : L^\infty(\mathbb{R}_+) \longrightarrow L^\infty(\mathbb{R}_+), \qquad u \longmapsto h \star u,$$

is bounded, i.e., $\mathrm{dom}\,\Lambda = L^\infty(\mathbb{R}_+)$, iff $\widehat{h} \in \widehat{\mathcal{A}}$ and:

$$\| \Lambda \|_{\mathcal{L}(L^\infty(\mathbb{R}_+), L^\infty(\mathbb{R}_+))} = \sup_{0 \neq u \in L^\infty(\mathbb{R}_+)} \frac{\| h \star u \|_\infty}{\| u \|_\infty} = \| \hat{h} \|_{\widehat{\mathcal{A}}}.$$

3. $\hat{f} \in \widehat{\mathcal{A}}$ is analytic and bounded in $\overline{\mathbb{C}_+} = \{s \in \mathbb{C} \mid \mathrm{Re}\, s \geq 0\}$ and continuous on $i\,\mathbb{R}$:

$$\| \widehat{f} \|_\infty \leq \| \widehat{f} \|_{\widehat{\mathcal{A}}}, \quad \widehat{\mathcal{A}} \subset H^\infty(\mathbb{C}_+) \quad (e^{-\frac{1}{s}} \in H^\infty(\mathbb{C}_+) \setminus \widehat{\mathcal{A}}).$$

4. BIBO-stability $\Rightarrow$ $L^p(\mathbb{R}_+) - L^p(\mathbb{R}_+)$-stability.

Example

- $p = \dfrac{e^{-h\,s}}{s-1} \notin H^\infty(\mathbb{C}_+)$ as p has a pole at $1 \in \mathbb{C}_+$,

$$\Rightarrow \Lambda : H^2(\mathbb{C}_+) \longrightarrow H^2(\mathbb{C}_+), \quad \widehat{u} \longmapsto \widehat{y} = \frac{e^{-h\,s}}{(s-1)}\,\widehat{u},$$

is unbounded,

$$\Rightarrow \lambda : L^2(\mathbb{R}_+) \longrightarrow L^2(\mathbb{R}_+), \quad u \longmapsto y = e^{t-h}\,Y \star u,$$

is unbounded.

- $u = e^{-t}\,Y \in L^2(\mathbb{R}_+)$: $\quad \| u \|_2 = \dfrac{1}{\sqrt{2}}, \quad \widehat{u} = \dfrac{1}{s+1} \in H^2(\mathbb{C}_+)$,

$$\Rightarrow \hat{y} = \frac{e^{-h\,s}}{s^2-1} \notin H^2(\mathbb{C}_+), \quad \frac{e^{-h\,s}}{s^2-1} = \mathcal{L}(\mathrm{sh}\,(t-h)\,Y).$$

$$\Rightarrow y(t) = \int_0^{t-h} e^{t-h-\tau}\,e^{-\tau}\,d\tau = \mathrm{sh}\,(t-h)\,Y \notin L^2(\mathbb{R}_+).$$

Example

- $p = \dfrac{e^{-h\,s}}{s-1} \notin \widehat{\mathcal{A}}$ because p has a pole at $1 \in \mathbb{C}_+$.

$$e^{-t}\,Y \in L^{\infty}(\mathbb{R}_+), \;\| \, e^{-t}\,Y \, \|_{\infty} = 1$$

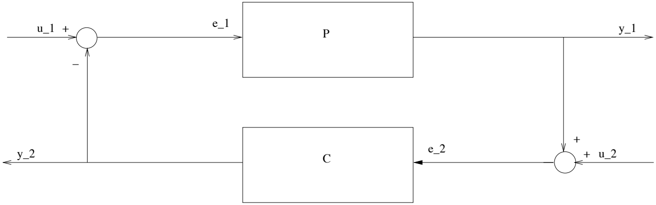
$$\Rightarrow y(t) = \int_0^{t-h} e^{t-h-\tau}\,e^{-\tau}\,d\tau = (\mathrm{sh}\,(t-h))\,Y \notin L^{\infty}(\mathbb{R}_+).$$

$\Rightarrow$ The plant p is not BIBO stable.

Stabilization problems

- Let the open-loop $\widehat{u} \longmapsto \widehat{y} = p\,\widehat{u}$ be unstable.

- Is it possible to find a controller c such that the closed-loop is stable, e.g., for all $\widehat{u}_1, \ \widehat{u}_2 \in H^2(\mathbb{C}_+)$ or for all $u_1, \ u_1 \in L^\infty(\mathbb{R}_+)$?



- Can we parametrize the set of stabilizing controllers of p?

- Is it possible to find robust/optimal controllers c of p?

The fractional representation of plants

• (Zames) The set of transfer functions has the structure of an algebra (parallel $+$, serie $\circ$, proportional feedback . by $\mathbb{R}$).

• (Vidyasagar) Let A be an algebra of stable transfer functions with a structure of an integral domain ($a\, b = 0,\ a \neq 0 \ \Rightarrow\ b = 0$) and $Q(A)$ its the field of fractions:

$$Q(A) = \{p = n/d \ \mid \ 0 \neq d,\ n \in A\}\,.$$

• K represents the class of systems:

$$p \in A \ \Rightarrow\ p \text{ is stable} \quad \& \quad p \in K \backslash A \ \Rightarrow\ p \text{ is unstable}.$$

• (Zames) The algebra A of stable transfer functions has to be a normed algebra so that we can consider the errors in the modelization & approximation of the real plant by a model

(e.g., Banach algebra: $\parallel a\, b \parallel_A \leq \parallel a \parallel_A \parallel b \parallel_A, \quad \parallel 1 \parallel_A = 1$).

Examples

- Let $RH_\infty = \mathbb{R}(s) \cap H^\infty(\mathbb{C}_+)$: algebra of exponentially-stable finite-dimensional plants:

$$RH_\infty = \{n/d \in \mathbb{R}(s) \mid \deg n \leq \deg d, \; d(\overline{s}) = 0 \Rightarrow \mathrm{Re}\, \overline{s} < 0\}.$$

$$p = \frac{1}{s-1} = \frac{\left(\frac{1}{s+1}\right)}{\left(\frac{s-1}{s+1}\right)}, \quad \frac{1}{s+1}, \quad \frac{s-1}{s+1} \in RH_\infty \; \Rightarrow \; p \in Q(RH_\infty).$$

- $\hat{\mathcal{A}}$: algebra of BIBO-stable ∞-dimensional plants:

$$p = \frac{e^{-h\,s}}{s-1} = \frac{\left(\frac{e^{-h\,s}}{s+1}\right)}{\left(\frac{s-1}{s+1}\right)}, \quad \frac{e^{-h\,s}}{s+1}, \quad \frac{s-1}{s+1} \in \hat{\mathcal{A}} \; \Rightarrow \; p \in Q(\hat{\mathcal{A}}).$$

- $H^\infty(\mathbb{C}_+)$: algebra of $L^2(\mathbb{R}_+)$-stable ∞-dimensional plants:

$$p = \frac{(1+e^{-2\,s})}{(1-e^{-2\,s})} \in Q(H^\infty(\mathbb{C}_+)) : \; 1+e^{-2\,s}, \; 1-e^{-2\,s} \in H^\infty(\mathbb{C}_+).$$

(Weakly) coprime factorization

- Let A be an algebra of stable transfer functions and:

$$K = Q(A) = \{n/d, \; 0 \neq d, \; n \in A\}.$$

- **Definition**: A transfer function $p \in K$ admits a weakly coprime factorization if:

$$\exists \, 0 \neq d, \, n \in A: \; p = n/d, \quad \forall \, k \in K: \; k\, n, \; k\, d \in A \; \Rightarrow \; k \in A.$$

- **Definition**: A transfer function $p \in K$ admits a coprime factorization over A if:

$$\exists \, 0 \neq d, \, n, \, x, \, y \in A: \quad p = n/d, \quad d\, x + n\, y = 1.$$

- A coprime factorization is a weakly coprime factorization:

$$\forall \, k \in K: \; k\, n, \; k\, d \in A \; \Rightarrow \; k = (k\, d)\, x + (k\, n)\, y \in A.$$

Examples

- **Example**: Let $A = RH_\infty$ and $p = \frac{1}{(s-1)} \in \mathbb{R}(s)$. Then,

$$p = \frac{n}{d}, \quad n = \frac{1}{(s+1)\,(s+2)}, \quad d = \frac{(s-1)}{(s+1)\,(s+2)} \in A,$$

is not a weakly coprime facorization as:

$$(s+2) \in Q(A) = \mathbb{R}(s), \quad (s+2) \notin A, \quad \left\{ \begin{array}{l} (s+2)\,n = \frac{1}{(s+1)} \in A, \\ (s+2)\,d = \frac{(s-1)}{(s+1)} \in A. \end{array} \right.$$

- **Example**: Let $A = RH_\infty$ and $p = \frac{1}{(s-1)} \in \mathbb{R}(s)$. Then,

$$p = \frac{n}{d}, \quad n = \frac{1}{(s+1)}, \quad d = \frac{(s-1)}{(s+1)} \in A,$$

is a coprime factorization of p as we have:

$$\frac{(s-1)}{(s+1)} + 2\,\frac{1}{(s+1)} = 1, \quad x = 1, \quad y = 2.$$

Internal stabilizability

- Let A be an algebra of stable transfer functions, $K = Q(A)$.
- Let $p \in K$ be a plant and $c \in K$ a controler.
- The closed-loop system is defined by:

$$\left(\begin{array}{c} u_1 \\ u_2 \end{array}\right) = \left(\begin{array}{cc} 1 & c \\ -p & 1 \end{array}\right) \left(\begin{array}{c} e_1 \\ e_2 \end{array}\right), \quad \left\{\begin{array}{l} y_1 = e_2 - u_2, \\ y_2 = u_1 - e_1. \end{array}\right.$$

- **Definition**: c internally stabilizes p if we have:

$$H(p,c) = \left(\begin{array}{cc} 1 & c \\ -p & 1 \end{array}\right)^{-1} = \left(\begin{array}{cc} \frac{1}{1+p\,c} & -\frac{c}{1+p\,c} \\ \frac{p}{1-p\,c} & \frac{1}{1+p\,c} \end{array}\right) \in A^{2\times 2}.$$

Then, c is called a stabilizing controler of p.

Examples

- **Example**: $A = RH_\infty, \quad K = Q(A) = \mathbb{R}(s)$.

$$\left\{ \begin{array}{l} p = \frac{s}{(s-1)}, \\ c = \frac{(s-1)}{(s+1)}, \end{array} \right. \Rightarrow \left\{ \begin{array}{l} e_1 = \frac{(s+1)}{(2s+1)} \, u_1 + \frac{(-s+1)}{(2s+1)} \, u_2, \\ e_2 = \frac{s(s+1)}{(2s+1)(s-1)} \, u_1 + \frac{(s+1)}{(2s+1)} \, u_2. \end{array} \right.$$

$\Rightarrow$ c does not internally stabilize p because:

$$\frac{s(s+1)}{(2s+1)(s-1)} \notin RH_\infty \quad (\text{pole at } 1 \in \mathbb{C}_+).$$

- **Example**: $A = RH_\infty, \quad K = Q(A) = \mathbb{R}(s)$.

$$\left\{ \begin{array}{l} p = \frac{s}{(s-1)}, \\ c = -2, \end{array} \right. \Rightarrow \left\{ \begin{array}{l} e_1 = -\frac{(s-1)}{(s+1)} \, u_1 - 2 \, \frac{(s-1)}{(s+1)} \, u_1, \\ e_2 = -\frac{s}{(s+1)} \, u_2 - \frac{(s-1)}{(s+1)} \, u_2. \end{array} \right.$$

$\Rightarrow$ the controller c internally stabilizes the plant p.

Strong and simultaneous stabilizabilities

- Let A be an algebra of stable transfer functions, $K = Q(A)$.

- **Definition**: $p \in K$ is strongly stabilizable if there exists a stable controller c, i.e., $c \in A$, which internally stabilizes p.

- **Definition**: The plants $p_1, \ldots, p_n \in K$ are simultaneously stabilizable if $\exists\ c \in K$ which internally stabilizes $p_1, \ldots, p_n$.

- Interests of the strong stabilizability:

Safety, good ability to track reference inputs.

- Interests of the simultaneous stabilizability:

The controller is designed to stabilize a family of plants, e.g.:

operating conditions, failed modes, loss of sensors/actuators, changes of operating points.

Examples

- **Example**: $A = RH_\infty$. The plant $p = \frac{1}{(s-1)} \in Q(A)$ is strongly stabilized by $c = 2$ as we have:

$$\frac{1}{1+p\,c} = \frac{(s-1)}{(s+1)}, \quad \frac{p}{1+p\,c} = \frac{1}{(s+1)}, \quad \frac{c}{1+p\,c} = \frac{2\,(s-1)}{(s+1)}.$$

- **Example**: $A = H^\infty(\mathbb{C}_+)$. The plant $p = \frac{(1+e^{-2\,s})}{(1-e^{-2\,s})} \in Q(A)$ is strongly stabilized by $c = 1$ as we have:

$$\frac{1}{1+p\,c} = \frac{1-e^{-2\,s}}{2}, \quad \frac{p}{1+p\,c} = \frac{1+e^{-2\,s}}{2}, \quad \frac{c}{1+p\,c} = \frac{1-e^{-2\,s}}{2}.$$

- **Example**: $A = RH_\infty$. The rational plants defined by

$$p_1 = \frac{1}{(s+1)}, \quad p_2 = \frac{2\,s}{(s-1)\,(s+1)},$$

are simultaneously stabilized by $c = 2\,\frac{(s+1)}{(s-1)}$.

Robust stabilizability

- Let A be a Banach algebra of stable transfer functions

$$(\text{e.g., } A = H^\infty(\mathbb{C}_+),\ \widehat{\mathcal{A}},\ A(\mathbb{D}),\ W_+).$$

- **Definition**: Let $c \in K = Q(A)$ be a stabilizing controller of $p \in K$. Then, c robustly stabilizes p if there exits $\epsilon > 0$ such that c internally stabilizes one of the family of plants:

1. Additive perturbations:

$$B_1(p, \delta) = \{p + \delta \mid \forall\, \delta \in A,\ \parallel \delta \parallel_A < \ \epsilon\}.$$

2. Multiplicative perturbations:

$$B_2(p, \delta) = \{p/(1 + \delta\, p) \mid \forall\, \delta \in A,\ \parallel \delta \parallel_A < \ \epsilon\}.$$

3. Relative additive perturbations:

$$B_3(p, \delta) = \{p\,(1 + \delta) \mid \forall\, \delta \in A,\ \parallel \delta \parallel_A < \ \epsilon\}.$$

4. Relative multiplicative perturbations:

$$B_4(p, \delta) = \{p/(1 + \delta) \mid \forall\, \delta \in A,\ \parallel \delta \parallel_A < \ \epsilon\}.$$

A fractional ideal approach (SCL 03)

- A is an integral domain of SISO stable plants and $K = Q(A)$.
- Let $p \in K$ be a plant and let us introduce the fractional ideal:

$$J = (1,\, p) \triangleq A + A\, p.$$

- J is defined by all the stable linear combinations of 1 and p.
- Why do we need 1? Algebraic answer: the structural properties of a plant p only depend on the system:

$$y - p\, u = 0 \;\Leftrightarrow\; (1 \quad -p) \begin{pmatrix} y \\ u \end{pmatrix} = 0.$$

Analysis answer: the structural properties of a plant p depend on the graph of the unbounded operator:

$$u \longmapsto y = p\, u.$$

Theory of fractional ideals

"Dedekind's invention of ideals in the 1870s was a major turning point in the development of algebra", Stillvell.

- **Definition:** An A-submodule J of $K = Q(A)$ is a fractional ideal of A if $\exists\, 0 \neq d \in A$ such that $(d)\, J = \{d\, j \mid j \in J\} \subseteq A$.

- A fractional ideal $J \subseteq A$ is called an ideal of A.

- A fractional ideal J is principal if $\exists\, k \in K$ s.t. $J = A\, k = (k)$.

- $I\, J = \{\sum_{i=1}^{n} a_i\, b_i \mid a_i \in I,\; b_i \in J\}$, $A : J = \{k \in K \mid (k)\, J \subseteq A\}$.

- A fractional ideal J is invertible if $\exists\, I \in \mathcal{F}(A)$ such that $I\, J = A$.

If J is invertible then its inverse J^{-1} is unique and defined by $A : J$.

Main results (SCL 03)

- Let A be a ring of stable transfer functions and $K = Q(A)$.
- Let $p \in K$ be a transfer function.
- Let $J = (1,\, p)$ be a fractional ideal, $A : J = \{d \in A \mid d\ p \in A\}$.
- **Theorem**: 1. p is stable iff $J = A$ iff $A : J = A$.

2. p admits a weakly coprime factorization iff:

$$\exists\, 0 \neq d \in A : \quad A : J = (d).$$

Then, $p = n/d$, $(n = d\, p \in A)$, is a weakly coprime factorization.

3. p is internally stabilizable iff J is invertible, i.e., iff:

$$\exists\, a,\, b \in A, \quad a + b\, p = 1, \quad a\, p \in A.$$

If $a \neq 0$, then $c = b/a$ is a stabilizing controller of p and:

$$J^{-1} = (a,\ b), \quad a = 1/(1 + p\, c), \quad b = c/(1 + p\, c).$$

Main results (SCL 03)

4. $c \in K$ internally stabilizes $p \in K$ if we have:

$$(1,\ p)\,(1,\ c) = (1 + p\,c).$$

5. $c \in K$ externally stabilizes $p \in K$, i.e., $p\,c/(1 - p\,c) \in A$, iff:

$$(1,\ p\,c) = (1 + p\,c).$$

6. p is strongly stabilizable iff there exists $c \in A$ such that:

$$(1,\ p) = (1 + p\,c).$$

7. p admits a coprime factorization iff J is principal.

Then, there exists $0 \neq d \in A$ such that $(1,\ p) = (1/d)$ and $p = n/d$ is a coprime factorization of p ($n = d\,p \in A$).

Proof 1

- Let $p \in K$ and $J = (1,\ p)$. If J is invertible, then we have:

$$1 \in J\,(A : J) = (1,\ p)\,(\{d \in A \mid d\,p \in A\}) = \{\alpha + \beta\,p \mid \alpha,\ \beta \in A : J\}$$

$$\Leftrightarrow\ \exists\ a,\ b \in A : \begin{cases} a + b\,p = 1, \\ a\,p \in A, \quad b\,p \in A. \end{cases}$$

If $a \neq 0$, then $c = b/a \in K$ satisfies:

$$H(p,\ c) = \begin{pmatrix} \frac{1}{1+p\,c} & -\frac{c}{1+p\,c} \\ \frac{p}{1+p\,c} & \frac{1}{1+p\,c} \end{pmatrix} = \begin{pmatrix} a & -b \\ a\,p & a \end{pmatrix} \in A^{2 \times 2},$$

$\Rightarrow c = b/a$ internally stabilizes p $\quad (a = 0 \Rightarrow c = 1 + b$ IS $p)$.

- If p is internally stabilizable, then there exists $c \in K$ s.t.:

$$a = 1/(1+p\,c) \in A, \quad a\,p = p/(1+p\,c) \in A, \quad b = c/(1+p\,c) \in A.$$

Let $I = (a,\ b)$. Then, $a + b\,p = 1 \in I\,J \Rightarrow I\,J = A \Rightarrow I = J^{-1}$.

Example

- $A = H^\infty(\mathbb{C}_+)$, $\quad p = \frac{e^{-s}}{(s-1)} \in Q(A)$, $\quad J = (1,\, p)$.

$$\gcd\left(\frac{e^{-s}}{s+1},\, \frac{s-1}{s+1}\right) = 1 \;\Rightarrow\; A : J = \{d \in A \mid d\, p \in A\} = \left(\frac{s-1}{s+1}\right).$$

- p is internally stabilizable iff

$$\exists\, a,\, b \in (A : J) :\; a + b\, p = 1 \quad \Leftrightarrow \quad \exists\, x,\, y \in A : \begin{cases} a = \left(\frac{s-1}{s+1}\right) x, \\ b = \left(\frac{s-1}{s+1}\right) y, \\ a + b\, p = 1. \end{cases}$$

$$a + b\, p = 1 \;\Leftrightarrow\; \left(\frac{s-1}{s+1}\right)(x + p\, y) = 1 \;\Leftrightarrow x = \frac{s+1}{s-1} - p\, y$$

$$\Leftrightarrow x = \frac{(s+1) - e^{-s}\, y}{s-1}.$$

- $x \in A \;\Rightarrow\; ((s+1) - e^{-s}\, y(s))(1) = 0 \;\Rightarrow\; y(1) = 2\, e.$

Example

- Hence, we can take:

$$y(s) = 2\,e \;\Rightarrow\; x(s) = 1 + 2\left(\frac{1 - e^{-(s-1)}}{s-1}\right) \in A.$$

- Therefore, we get:

$$\begin{cases} a = \left(\frac{s-1}{s+1}\right) x = \left(\frac{s-1}{s+1}\right) \left(1 + 2\left(\frac{1-e^{-(s-1)}}{s-1}\right)\right), \\ b = \left(\frac{s-1}{s+1}\right) y = 2\,e\left(\frac{s-1}{s+1}\right), \\ a + b\,p = 1. \end{cases}$$

$\Rightarrow$ A stabilizing controller c of p is defined by:

$$c = \frac{b}{a} = \frac{2\,e\,(s-1)}{(s-1) + 2\,(1 - e^{-(s-1)})} = \frac{2\,e\,(s-1)}{s + 1 - 2\,e^{-(s-1)}}.$$

Proof 2

- $J = (1,\ p)$ is principal iff there exists $0 \neq k \in K$ s.t. $J = (k)$, i.e., iff there exist $0 \neq d,\ n,\ x,\ y \in A$ s.t.:

$$\left\{ \begin{array}{l} 1 = d\,k, \\ p = n\,k, \\ k = x + y\,p \end{array} \right. \Leftrightarrow \left\{ \begin{array}{l} k = 1/d, \\ p = n/d, \\ 1/d = x + y\,(n/d), \end{array} \right. \Leftrightarrow \left\{ \begin{array}{l} p = n/d, \\ d\,x + n\,y = 1. \end{array} \right.$$

p admits a coprime factorization $p = n/d$ iff $J = (1/d)$.

- A principal fractional ideal $J = (k)$ is invertible: $J^{-1} = (1/k)$.

- $(d\,x) + (d\,y)\,p = 1$, i.e., $a = d\,x,\ b = d\,y \in J^{-1} = (d)$,

$$\Rightarrow c = b/a = y/x \in \mathrm{Stab}(p).$$

Example

- Let $A = H^{\infty}(\mathbb{C}_+)$ and $p = \frac{e^{-s}}{(s-1)} \in K = Q(A)$.
- Let $J = (1,\, p)$ be the fractional ideal of A generated by 1 and p.
- $J = \left(\frac{s+1}{s-1}\right)$ is a principal ideal as we have:

$$\begin{cases} 1 = \left(\frac{s-1}{s+1}\right) \left(\frac{s+1}{s-1}\right), \\ \frac{e^{-s}}{(s-1)} = \left(\frac{e^{-s}}{s+1}\right) \left(\frac{s+1}{s-1}\right), \\ \frac{(s+1)}{(s-1)} = \left(1 + 2\left(\frac{1-e^{-(s-1)}}{s-1}\right)\right) + 2\, e\, \frac{e^{-s}}{(s-1)} \qquad (\star). \end{cases}$$

$p = \frac{n}{d}$, $n = \frac{e^{-s}}{(s+1)}$, $d = \frac{(s-1)}{(s+1)}$, is a coprime factorization of p:

$$(\star) \Leftrightarrow \left(\frac{s-1}{s+1}\right) \left(1 + 2\left(\frac{1-e^{-(s-1)}}{s-1}\right)\right) + \left(\frac{e^{-s}}{s+1}\right) (2\, e) = 1.$$

Proof 3

- If p is strongly stabilizable then there exists $c \in A$ such that:

$$a = \frac{1}{1 + p\,c} \in A, \quad a\,p = \frac{p}{1 + p\,c} \in A, \quad b = \frac{c}{1 + p\,c} = c\,a \in A.$$

Using the fact that $c \in A$, we obtain:

$$J^{-1} = (a,\ b) = (a) = ((1 + p\,c)^{-1}) \ \Rightarrow\ J = (J^{-1})^{-1} = (1 + p\,c).$$

- We suppose that there exists $c \in A$ such that $(1,\ p) = (1 + p\,c)$

$$\Rightarrow\ \exists\, 0 \neq d,\ n \in A : \left\{ \begin{array}{l} 1 = d\,(1 + p\,c), \\ p = n\,(1 + p\,c), \end{array} \right. \ \Rightarrow \left\{ \begin{array}{l} p = n/d, \\ d + n\,c = 1, \end{array} \right.$$

$$\Rightarrow \left(\begin{array}{cc} \frac{1}{1+p\,c} & -\frac{c}{1+p\,c} \\ \frac{p}{1+p\,c} & \frac{1}{1+p\,c} \end{array} \right)^{-1} = \left(\begin{array}{cc} d & -d\,c \\ n & d \end{array} \right) \in A^{2\times 2}$$

$\Rightarrow$ $c \in A$ internally stabilizes p, i.e., p is strongly stabilizable.

Example

- $A = H^\infty(\mathbb{C}_+), \quad K = Q(A), \quad p = \frac{(1+e^{-2s})}{(1-e^{-2s})} \in K.$
- We have $J = (1,\, p) = \left(\frac{1}{1-e^{-2s}}\right)$ because:

$$\begin{cases} 1 = (1-e^{-2s})\,\dfrac{1}{(1-e^{-2s})}, \\ p = \dfrac{(1+e^{-2s})}{(1-e^{-2s})} = (1+e^{-2s})\,\dfrac{1}{(1-e^{-2s})}, \\ \dfrac{1}{(1-e^{-2s})} = \dfrac{1}{2} + \dfrac{1}{2}\,\dfrac{(1+e^{-2s})}{(1-e^{-2s})}. \end{cases}$$

$\Rightarrow$ coprime factorization $\begin{cases} p = \dfrac{(1+e^{-2s})}{(1-e^{-2s})}, \\ \frac{1}{2}\,(1-e^{-2s}) + \frac{1}{2}\,(1+e^{-2s}) = 1. \end{cases}$

$\Rightarrow$ $c = 1$ is a stable stabilizing controller of p.

- **Check:** $1 + p\,c = \frac{2}{(1-e^{-2s})}$ and $J = \left(\frac{1}{1-e^{-2s}}\right) = (1+p\,c)$.

Example

- Let A be the Banach algebra W_+ of analytic functions in the unit disc $\mathbb{D}$ whose Taylor series converge absolutely:

$$W_+ = \left\{ f(z) = \sum_{i=0}^{+\infty} a_i\, z^i \mid \sum_{i=0}^{+\infty} |a_i| < +\infty \right\}.$$

- A is the algebra of the BIBO-stable causal filters.

- Let us consider $J = (1,\, p)$ where $p = e^{-\left(\frac{1+z}{1-z}\right)}$

$$\begin{cases} n = (1-z)^3\, e^{-\left(\frac{1+z}{1-z}\right)} \in A, \\ d = (1-z)^3 \in A, \end{cases} \quad \Rightarrow p = n/d \in Q(A).$$

- The ideal $A : J = \{d \in A \mid d\, p \in A\}$ of A is not finitely generated (R. Mortini & M. Von Renteln, "Ideals in Wiener algebra", J. Austral. Math. Soc., 46 (1989), 220-228).

$\Rightarrow$ p does not admit a (weakly) coprime factorization and is not internally stabilizable.

Example

• Let A be the disc algebra $A(\mathbb{D})$ of holomorphic functions in the open unit disc $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$ which are continuous on the unit circle $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$.

• We have $\quad n = (1 - z)\, e^{-\left(\frac{1+z}{1-z}\right)} \in A, \quad d = (1 - z) \in A,$

$$\Rightarrow p = n/d = e^{-\left(\frac{1+z}{1-z}\right)} \in Q(A), \quad J = (1,\, p).$$

• The ideal $A : J = \{d \in A \mid d\, p \in A\} = \{d \in A \mid d(1) = 0\}$ of A is maximal and is not finitely generated (R. Mortini, "Finitely generated prime ideals in H^∞ and $A(\mathbb{D})$", Math. Z., 191 (1986), 297-302).

$\Rightarrow$ p does not admit a (weakly) coprime factorization and is not internally stabilizable.

Robust stabilization

- Let $p \in Q(A)$ and $\delta \in A$.
- $c \in Q(A)$ internally stabilizes p and $p + \delta$ iff we have:

$$\left\{\begin{array}{l} (1,\ p)\,(1,\ c) = (1 + p\,c), \\ (1,\ p + \delta)\,(1,\ c) = (1 + (p + \delta)\,c), \end{array}\right. \Leftrightarrow \left\{\begin{array}{l} (1,\ p)\,(1,\ c) = (1 + p\,c), \\ (1,\ p)\,(1,\ c) = (1 + (p + \delta)\,c), \end{array}\right.$$

$$\Leftrightarrow \left\{\begin{array}{l} (1,\ p)\,(1,\ c) = (1 + p\,c), \\ \left(\dfrac{1 + (p + \delta)\,c}{1 + p\,c}\right) = \left(1 + \dfrac{\delta\,c}{1 + p\,c}\right) = A, \end{array}\right. \Leftrightarrow \left\{\begin{array}{l} c \text{ IS } p, \\ 1 + \dfrac{\delta\,c}{(1 + p\,c)} \in \mathrm{U}(A). \end{array}\right.$$

- If A is a Banach algebra, then (small gain theorem):

$$\|\ 1 - a\ \|_A < \ 1 \ \Rightarrow \ a \in \mathrm{U}(A) = \{a \in A \mid \exists\, b \in A :\ a\,b = b\,a = 1\}.$$

$\Rightarrow$ A sufficient condition for robust stabilization is:

$$\|\ \delta\ \|_A < \|\ c/(1 + p\,c)\ \|_A^{-1}\ .$$

Robust stabilization

- Let A be a Banach algebra, $p \in Q(A)$ and $\delta \in A$.
- c internally stabilizes p and $p/(1+\delta\, p)$ iff we have:

$$\begin{cases} (1, \; p)\,(1, \; c) = (1 + p\, c), \\ \left(1, \; \frac{p}{(1+\delta\, p)}\right)\,(1, \; c) = \left(1 + \frac{p\, c}{(1+\delta\, p)}\right), \end{cases}$$

$$\Leftrightarrow \begin{cases} (1, \; p)\,(1, \; c) = (1 + p\, c), \\ (1 + \delta\, p, \; p)\,(1, \; c) = (1 + p\, c + \delta\, p), \end{cases}$$

$$\Leftrightarrow \begin{cases} (1, \; p)\,(1, \; c) = (1 + p\, c), \\ \left(\dfrac{1 + p\, c + \delta\, p}{1 + p\, c}\right) = \left(1 + \dfrac{\delta\, p}{1 + p\, c}\right) = A, \end{cases} \Leftrightarrow \begin{cases} c \text{ IS } p, \\ 1 + \dfrac{\delta\, p}{(1 + p\, c)} \in \mathrm{U}(A). \end{cases}$$

$\Rightarrow$ A sufficient condition for robust stabilization is:

$$\| \, \delta \, \|_A < \| \, p/(1 + p\, c) \, \|_A^{-1} \, .$$

General Q-parametrization (SCL 03)

- **Theorem**: Let c be a stabilizing controller of $p \in Q(A)$,

$$a = 1/(1 + p\,c), \quad b = c/(1 + p\,c), \quad J = (1,\, p).$$

Then, all stabilizing controllers of p are

$$c(q_1,\, q_2) = \frac{b + a^2\, q_1 + b^2\, q_2}{a - a^2\, p\, q_1 - b^2\, p\, q_2}, \quad (\star)$$

where q_1 and q_2 any element of A: $a - a^2\, p\, q_1 - b^2\, p\, q_2 \neq 0$.

1. $(\star)$ depends on only one free parameter

$$\Leftrightarrow \; p^2 \text{ admits a coprime factorization } p^2 = s/r.$$

$$(\star) \;\Leftrightarrow\; c(q) = \frac{b + r\, q}{a - r\, p\, q}, \quad \forall\, q \in A: \; a - r\, p\, q \neq 0.$$

2. If p admits a coprime factorization $p = n/d$, $d\,x + n\,y = 1$:

$$(\star) \;\Leftrightarrow\; c(q) = \frac{y + d\, q}{x - n\, q}, \quad \forall\, q \in A: \; x - n\, q \neq 0.$$

Solving Zames-Francis' conditions

- Let $p \in Q(A)$ be an internally stabilizable plant

$$\Leftrightarrow \; \exists \, a, \, b \in A, \quad a + b\,p = 1, \quad a\,p \in A, \quad (\star)$$

$$\Leftrightarrow \; \exists \, b \in A, \quad b\,p, \quad p\,(1 + b\,p) \in A. \quad \text{(Zames-Francis)}$$

- If $a \neq 0$, then $c = b/a = b/(1 + b\,p)$ internally stabilizes p.

- $J = (1, \, p)$ is invertible and $J^{-1} = (a, \, b) \Rightarrow J^2 = (1, \, p, \, p^2)$

$$\Rightarrow J^{-2} = (J^2)^{-1} = \{\alpha \in A \,|\, \alpha\,p, \; \alpha\,p^2 \in A\}.$$

$$\Rightarrow J^{-2} = (J^{-1})^2 = (a, \, b)^2 = (a, \, a\,b, \, b^2).$$

- Using $(\star)$, we get $a\,b = (b)\,a^2 + (a\,p)\,b^2 \in (a^2, \, b^2)$

$$\Rightarrow J^{-2} = (a^2, a\,b, \, b^2) = (a^2, \, b^2).$$

Solving Zames-Francis' conditions

- Let us find all the possible a' and b' satisfying:

$$\exists\, a',\, b' \in A, \quad a' + b'\, p = 1, \quad a'\, p \in A, \quad (1)$$

- Using the fact that a, b, $a\,p \in A$ and $a + b\,p = 1$, we get:

$$(b' - b)\, p = a - a' \in A, \quad (b' - b)\, p^2 = (a - a')\, p \in A,$$

$$\Rightarrow\; b' - b \in \{\alpha \in A \,|\, \alpha\, p,\; \alpha\, p^2 \in A\} = (a^2,\, b^2),$$

$$\Rightarrow\; \exists\, q_1,\, q_2 \in A : \quad \begin{cases} b' = b + q_1\, a^2 + q_2\, b^2, \\ a' = a - (q_1\, a^2 + q_2\, b^2)\, p, \end{cases} \quad (2)$$

$$\Rightarrow\; c' = \frac{b'}{a'} = \frac{b + q_1\, a^2 + q_2\, b^2}{a - (q_1\, a^2 + q_2\, b^2)\, p} \in \mathrm{Stab}(p).$$

- We can check that, for all q_1 and $q_2 \in A$, (2) satisfies (1).

Zames-Francis Q-parametrization

- Let $p = n/d$ be a coprime factorization of p over A:

$$d\,x + n\,y = 1.$$

$$\Rightarrow \;\; J = (1/d) \Rightarrow J^{-2} = (d^2),$$

$$\Rightarrow \;\; a = d\,x, \;\; b = d\,y \in J^{-1} = (d): \;\; a + b\,p = 1, \quad a\,p \in A.$$

$$\Rightarrow \;\; c(q) = \frac{b + q\,d^2}{a - q\,d^2\,p} = \frac{d\,y + q\,d^2}{d\,x - d\,n\,q} = \frac{y + q\,d}{x - n\,q}.$$

- **Conclusion:** We have just found again Zames-Francis and Youla-Kučera parametrizations of all stabilizing controllers of p.

K. Mori, CDC 1999, 973-975

- Let $A = \mathbb{R}[x^2, x^3]$ be the ring of discrete time delay systems without the unit delay.
- A is used for high-speed circuits, computer memory devices...
- $p = (1 - x^3)/(1 - x^2) \in Q(A), \quad J = (1, p)$.
- Using $(1 - x^3)(1 + x^3) = (1 - x^2)(1 + x^2 + x^4)$, we get

$$p = \frac{(1 - x^3)}{(1 - x^2)} = \frac{(1 + x^2 + x^4)}{(1 + x^3)}.$$

$A : J = (1 - x^2, 1 + x^3)$ is not principal because $(x + 1) \notin A$.

$\Rightarrow$ p does not admit a (weakly) coprime factorization.

- $J\,(A:J) = (1-x^2,\, 1+x^3,\, 1-x^3,\, 1+x^2+x^4)$.
- $(1+x^3)/2 + (1-x^3)/2 = 1 \in J\,(A:J)$

$$\Rightarrow \begin{cases} a = (1+x^3)/2 \in A:J, \\ b = (1-x^2)/2 \in A:J, \\ a + b\,p = 1, \end{cases}$$

$\Rightarrow\ c = b/a = (1-x^2)/(1+x^3)$ internally stabilizes p.

- $J^{-1} = (1-x^2,\, 1+x^3) \Rightarrow J^{-2} = ((1-x^2)^2,\, (1+x^3)^2)$.
- $(x+1) \notin A \ \Rightarrow\ J^{-2}$ is not principal ideal of A.

$\Rightarrow$ All stabilizing controllers of p have the form

$$c(q_1, q_2) = \frac{2\,(1-x^2) + (1+x^3)^2\,q_1 + (1-x^2)^2\,q_2}{2\,(1+x^3) - (1+x^3)\,(1+x^2+x^4)\,q_1 - (1-x^2)\,(1-x^3)\,q_2},$$

for all $q_1,\ q_2 \in A$ such that the denominator exists.

Distributed delay

- $p = e^{-s}/(s-1)$ is stabilized by the controller (distributed delay):

$$c = 2\,e\,(s-1)/\left(s+1-2\,e^{-(s-1)}\right).$$

$$\begin{cases} a = \frac{1}{(1+p\,c)} = \frac{(s+1-2\,e^{-(s-1)})}{(s+1)} \in H^{\infty}(\mathbb{C}_+), \\ b = \frac{c}{1+p\,c} = \frac{2\,e\,(s-1)}{(s+1)} \in H^{\infty}(\mathbb{C}_+), \\ a\,p = \frac{p}{(1+p\,c)} = \frac{e^{-s}}{(s+1)}\,\frac{(s+1-2\,e^{-(s-1)})}{(s-1)} \in H^{\infty}(\mathbb{C}_+). \end{cases}$$

- We obtain that all stabilizing controllers of p have the form:

$$c(l) = \frac{2\,e + l\,\frac{(s-1)}{(s+1)}}{1+2\left(\frac{1-e^{-(s-1)}}{s-1}\right) - l\,\frac{e^{-s}}{(s+1)}}, \quad l = \left(1+2\left(\frac{1-e^{-(s-1)}}{s-1}\right)\right)^2 q_1 + 4\,e^2\,q_2.$$

$\Rightarrow$ the Youla-Kučera parametrization for the coprime factorization:

$$p = \frac{n}{d}, \; n = \frac{e^{-s}}{(s+1)}, \; d = \frac{(s-1)}{(s+1)}, \; d\left(1+2\left(\frac{1-e^{-(s-1)}}{s-1}\right)\right) + n\,(2\,e) = 1.$$

Convexity of $H(p, c)$

- Let $c_\star$ be a stabilizing controller of $p \in Q(A)$.
- All stabilizing controllers of p are parametrized by

$$c(q_1, q_2) = \frac{(1 + p\, c_*)\, c_* + q_1 + q_2\, {c_*}^2}{(1 + p\, c_*) - q_1\, p - q_2\, p\, {c_*}^2}$$

$$\forall\, q_1,\, q_2 \in A: \;(1 + p\, c_*) - q_1\, p - q_2\, p\, {c_*}^2 \neq 0.$$

- Then, the closed-loop system

$$\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{1+p\,c} & -\frac{c}{1+p\,c} \\ \frac{p}{1+p\,c} & \frac{1}{1+p\,c} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \text{ is affine/convex in the } q_i\text{'s}$$

$$\begin{pmatrix} \frac{1}{1+p\,c_*} - q_1\, \frac{p}{(1+p\,c_*)^2} - q_2\, \frac{p\,{c_*}^2}{(1+p\,c_*)^2} & -\frac{c_*}{1+p\,c_*} - q_1\, \frac{1}{(1+p\,c_*)^2} - q_2\, \frac{{c_*}^2}{(1+p\,c_*)^2} \\ \frac{p}{1+p\,c_*} - q_1\, \frac{p^2}{(1+p\,c_*)^2} - q_2\, \frac{(p\,c_*)^2}{(1+p\,c_*)^2} & \frac{1}{1+p\,c_*} - q_1\, \frac{p}{(1+p\,c_*)^2} - q_2\, \frac{p\,{c_*}^2}{(1+p\,c_*)^2} \end{pmatrix}$$

i.e., $\forall\, \lambda \in A: \quad H(p,\, c(\lambda\, q_1 + (1-\lambda)\, q_1',\, \lambda\, q_2 + (1-\lambda)\, q_2'))$

$$= \lambda\, H(p,\, c(q_1,\, q_2)) + (1-\lambda)\, H(p,\, c(q_1',\, q_2')).$$

Sensitivity minimization

- Let A be a Banach algebra (e.g., $H^\infty(\mathbb{C}_+)$, $\widehat{\mathcal{A}}$, $W_+ \ldots$).
- Let $c_\star$ be a stabilizing controller of $p \in Q(A)$ and:

$$a = 1/(1 + p\, c_\star), \quad b = c_\star/(1 + p\, c_\star) \in A.$$

- Let $w \in A$ be a weighted function. Then, we have:

$$\inf_{c \in \mathrm{Stab}(p)} \| \, w/(1+p\,c) \, \|_A = \inf_{q_1,\, q_2 \in A} \| \, w\,(a - a^2\, p\, q_1 - b^2\, p\, q_2) \, \|_A \quad (\star)$$

i.e., $(\star)$ is a convex problem in the q_i's!

- If $p = n/d$ is a coprime factorization $(x,\, y \in A, d\, x + n\, y = 1)$

$$\Rightarrow \; a - a^2\, p\, q_1 - b^2\, p\, q_2 = d\,(x - q\, n).$$

$$\forall\, q \in A, \quad \exists\, q_1,\ q_2 \in A: \quad q = x^2\, q_1 + y^2\, q_2, \quad \text{where:}$$

$$q_1 = d^2\,(1 + 2\, n\, y)\, q, \quad q_2 = n^2\,(1 + 2\, d\, x)\, q.$$

$$(\star) \;\Leftrightarrow\; \inf_{q \in A} \| \, w\, d\,(x - n\, q) \, \|_A \, .$$

Short bibliography

• R. F. Curtain, H. J. Zwart, *An Introduction to ∞-Dimensional Linear Systems Theory*, TAM 21, Springer, 1991.

• A. Q., "On a generalization of the Youla-Kučera parametrization. Part I: The fractional ideal approach to SISO systems", *Systems & Control Letters*, 50 (2003), 135-148.

• A. Q., "An algebraic interpretation to the operator-theoretic approach to stability. Part I: SISO systems", Acta Applicandæ Mathematicæ, 88 (2005), 1-45.

• A. Q., "A lattice approach to analysis and synthesis problems", *MCSS*, 18 (2006), 147-186.

• A. Q., "On a generalization of the Youla-Kučera parametrization. Part II: The lattice approach to MIMO systems", *MCSS*, 18 (2006), 199-235.

• M. Vidyasagar, *Control System Synthesis: A Factorization Approach*, MIT Press, 1985.