Approximate stabilization of bilinear Schrödinger equations

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Presentation outline

1. Bilinear Schrödinger equation
2. N-D decaying potential (M.M., ArXiv : 0805.0910)
3. 1D infinite potential well (K. Beauchard and M.M, ArXiv : 0801.1522v1)

Bilinear Schrödinger equation

$$
\begin{aligned}
i\frac{d}{dt}\Psi(t) &= (H_0 + u(t)H_1)\Psi(t) \\
\Psi(0) &= \Psi_0
\end{aligned}
$$

$(\hbar = 1)$
$\Psi(t) \in \mathcal{H}$: System's wavefunction in appropriate Hilbert space,
H_0, H_1 : Hermitian operators on $\mathcal{H}$,
H_0 internal free Hamiltonian and H_1 the interaction Hamiltonian,
$u(t) \in \mathbb{R}$: Scalar control (Laser amplitude for instance).

$$
\|\Psi_0\|_{\mathcal{H}} = 1 \Longrightarrow \|\Psi(t)\|_{\mathcal{H}} = 1 \quad \forall t > 0.
$$

Infinite dimensional case : few results

G. Turinici 2000

H_1 bounded operator on $H^2_x(\mathbb{R}^N)$ and H_0 generating a C^0-semigroup of bounded linear operators on $H^2_x(\mathbb{R}^N)$

$$\longrightarrow$$

the complement of the attainable set, applying L^2-control fields, is everywhere dense in $\mathbb{S} \cap H^2_x(\mathbb{R}^N)$.

Th. Chambrion et al., 2008

H_0 admitting a discrete spectrum, the eigenvalue differences $\lambda_{j+1} - \lambda_j$ are $\mathbb{Q}$-linearly independent and $\langle H_1\phi_j, \phi_{j+1}\rangle \neq 0$

$$\longrightarrow$$

the system is approximately controllable.

Infinite dimensional case : stabilization

2 important test cases

$$i\frac{d}{dt}\Psi = -\triangle\Psi + V(x)\Psi + u(t)\mu(x)\Psi.$$

Decaying potential : $V(x) \to 0$ as $|x| \to \infty$ with a certain rate.

Infinite potential well : $V(x) = \begin{cases} 0 & \text{if} \quad x \in B_1(0), \\ \infty & \text{else.} \end{cases}$

Controllability :

Decaying potential : Controllability to be studied !

Infinite potential well :

K. Beauchard 2005, K. Beauchard and J.M. Coron 2006

For the 1D case and where $\mu(x) = x$ (moving potential well) : the local controllability (in H^7) and the controllability between the bound states.

V. Nersesyan 2008

For the 1D case and where $\mu(x) = x$ (moving potential well) : the exact controllability (in H^7).

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N-D decaying potential

$$i\frac{d}{dt}\Psi(t,x) = -\triangle\Psi(t,x) + V(x)\Psi(t,x) + u(t)\,\mu(x)\Psi(t,x), \quad (t,x)\in\mathbb{R}^+\times\mathbb{R}^N,$$
$$\Psi(0,x) = \Psi_0(x), \qquad \|\Psi_0\|_{L^2} = 1.$$

Decay assumption **(A)**

- $N = 1 : (1+|x|)V \in L^1(\mathbb{R})$;
- $N = 2 : |V(x)| \leq C(1+|x|)^{-3-\epsilon}$;
- $N = 3 : V \in L^{\frac{3}{2}-\epsilon}(\mathbb{R}^3) \cap L^{\frac{3}{2}+\epsilon}(\mathbb{R}^3)$;
- $N \geq 4 : \widehat{V} \in L^1$ and $(1+|x|^2)^{\gamma/2}V(x)$ is a bounded operator on the sobolev space H^ν for some $\nu > 0$ and $\gamma > N+4$.

Interaction Hamiltonian **(B)**

$\mu \in \mathcal{L}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ where $\mathcal{L} = \bigcup_{p\geq 2} L^p(\mathbb{R}^N)$.

Spectral properties

Spectra of $H_0 = -\triangle + V(x)$:

$$\sigma(H_0) = \sigma_d(H_0) \cup \sigma_c(H_0),$$

$\sigma_c(H_0) = [0, +\infty)$ absolutely continuous spectrum (absence of singular spectrum) and $\sigma_d(H_0) \subset (-\infty, 0)$ the bound states.

Discrete eigenspace :

$$\mathcal{E}_d = \text{span}\{\phi_i; i = 0, 1, 2, ..., M\}, \qquad \sigma_d(H_0) = \{\lambda_0, \lambda_1, ..., \lambda_M\}.$$

Free Dynamics

Initial state :

$$\Psi_0 = \Psi_{0,c} + \Psi_{0,d}.$$

$S(t)$: C_0-semigroup spanned by $(-\triangle + V(x))/i$.

Discrete part :

$$\Psi_{0,d} = \sum_{i=0}^{M} \alpha_i \phi_i(x) \qquad \Rightarrow \qquad \Pi_d S(t)\Psi_0 = \sum_{i=0}^{M} \alpha_i e^{-i\lambda_i t} \phi_i(x).$$

Absolutely continuous part : dispersive behavior.

Dispersive estimates

Theorem (dispersive estimate)

Under the decay assumption **(A)** on the potential V, we have

$$\|S(t)\mathbb{P}_{\rm ac}\|_{1\to\infty} \leq |t|^{-\frac{N}{2}}.$$

Corollary (interpolation)

$$\sup_{t>0} |t|^{N\left(\frac{1}{2}-\frac{1}{p}\right)} \|S(t)\mathbb{P}_{\rm ac}\psi\|_{L^{p'}} \leq \|\psi\|_{L^p} \quad \text{for all } \psi \in L^1(\mathbb{R}^N)\cap L^2(\mathbb{R}^N),$$

where $1 \leq p \leq 2$ and $\frac{1}{p}+\frac{1}{p'}=1$.

Strichartz inequality

$$\|S(t)\mathbb{P}_{\rm ac}\psi\|_{L^q_t(L^p_x)} \leq C\|\psi\|_{L^2}, \text{ for all } \frac{2}{q}+\frac{N}{p}=\frac{N}{2}, \quad 2<q\leq\infty.$$

Controlled system : a first approach

Similarly to the finite dimensional case (Mirrahimi, Rouchon, Turinici, Automatica : 2005)

$$i\frac{d}{dt}\Psi(t) = (-\triangle + V(x) + u(t)\mu(x))\Psi(t), \qquad \Psi(0) = \Psi_0.$$

Target state : ϕ_0 a bound state of $H_0 = -\triangle + V(x)$.

Lyapunov function :

$$\mathcal{V}(\Psi) = 1 - |\langle \Psi, \phi_0 \rangle|^2$$

Thus :

$$\frac{d\mathcal{V}}{dt} = -u(t)\Im(\langle \mu\Psi(t), \phi_0 \rangle \langle \phi_0, \Psi(t) \rangle).$$

Feedback :

$$u = c\,\Im(\langle \mu\Psi(t), \phi_0 \rangle \langle \phi_0, \Psi(t) \rangle), \qquad c > 0$$

LaSalle in finite dimensions

$$\frac{d}{dt}X = f(X), \qquad \frac{d}{dt}\mathcal{V}(X) \leq 0.$$

- $0 \leq \mathcal{V}(X) \searrow \alpha$.
- $\mathcal{V}$ being continuous and radially unbounded, X is bounded and so $X(t_n) \to \overline{X}$.
- $\mathcal{V}$ continuous and so $\mathcal{V}(\overline{X}) = \alpha$.
- We consider $\widetilde{X}$ the solution of the system starting at $\overline{X}$.
- The flow being continuous with respect to the initial state $X(t_n + \tau) \to \widetilde{X}(\tau)$.
- So $\mathcal{V}(\widetilde{X}(\tau)) = \mathcal{V}(\widetilde{X}(0)) = \alpha$ and therefore the LaSalle invariance principle.

Convergence analysis

Main obstacle

The pre-compactness of trajectories in L^2 even in the case where the system is initialized in the finite dimensional eigenspace spanned by the bound states.

We can have phenomena such as the L^2-Mass lost at infinity when crossing the continuous part of the spectrum.

Idea

$$\frac{d}{dt}X = f(X), \qquad \frac{d}{dt}\mathcal{V}(X) \leq 0.$$

- $0 \leq \mathcal{V}(X) \searrow \alpha$.
- $\mathcal{V}$ being continuous and radially unbounded, X is bounded and so $X(t_n) \rightharpoonup \overline{X}$ in a weak sense.
- If $\mathcal{V}$ continuous with respect to this weak topology, $\mathcal{V}(\overline{X}) = \alpha$.
- We consider $\widetilde{X}$ the solution of the system starting at $\overline{X}$.
- If the flow is continuous with respect to the initial state for this weak topology, $X(t_n + \tau) \rightharpoonup \widetilde{X}(\tau)$.
- So $\mathcal{V}(\widetilde{X}(\tau)) = \mathcal{V}(\widetilde{X}(0)) = \alpha$ and therefore the LaSalle invariance principle.
- Note that this only characterizes a weak ω-limit set and in order to show that we have an approximate strong convergence we need to ensure that only a small part of the mass is lost during the weak convergence.

Main result

Theorem (approximate stabilization)

Consider ($V(x)$ and $\mu(x)$ satisfying **A** and **B**)

$$i\frac{d}{dt}\Psi(t,x) = -\triangle\Psi(t,x) + V(x)\Psi(t,x) + u(t)\mu(x)\Psi(t,x), \qquad \Psi|_{t=0} = \Psi_0(x).$$

Assume :

- A1 $\Psi_0 = \sum_{i=0}^{M} \alpha_i \phi_i$, $\{\phi_i\}_{i=0}^{M}$ bound states of $-\triangle + V(x)$;
- A2 $\alpha_0 \neq 0$;
- A3 Non-degenerate transitions : $\lambda_{i_1} - \lambda_{j_1} \neq \lambda_{i_2} - \lambda_{j_2}$ for $(i_1, j_1) \neq (i_2, j_2)$;
- A4 Simple (mono-photonic) transitions : $\langle \mu\phi_i, \phi_j \rangle \neq 0 \quad i \neq j$.

Then for all $\epsilon > 0$, there exists a feedback law of the form

$$u(t) = u_{\epsilon,\alpha}(\Psi(t)) = c\left|f_\epsilon(\Psi(t))\right|^\alpha f_\epsilon(\Psi(t)), \qquad c > 0, \alpha \geq 0,$$

$$f_\epsilon = [(1-\epsilon)\sum_{i=0}^{M} \Im(\langle \mu\Psi, \phi_i\rangle \langle \phi_i, \Psi\rangle) + \epsilon\Im(\langle \mu\Psi, \phi_0\rangle \langle \phi_0, \Psi\rangle)]$$

the system admits a unique global strong solution. Furthermore

$$\liminf_{t\to\infty} |\langle \Psi(t,x), \phi_0(x)\rangle|^2 > 1 - \epsilon.$$

Feedback law

Lyapunov function :

$$\mathcal{V}_\epsilon(\Psi) = 1 - (1-\epsilon)\sum_{i=0}^{M} |\langle \Psi, \phi_i \rangle|^2 - \epsilon |\langle \Psi, \phi_0 \rangle|^2.$$

We have,

$$0 \leq \mathcal{V}_\epsilon(\Psi) \qquad \text{and} \qquad \mathcal{V}_\epsilon(\Psi) = 0 \Leftrightarrow |\langle \Psi, \phi_0 \rangle| = 1,$$

moreover,

$$\mathcal{V}_\epsilon(\Psi_0) \leq \epsilon.$$

Then,

$$\frac{d}{dt}\mathcal{V}_\epsilon(\Psi) = -u(t) f_\epsilon(\Psi(t)),$$

which implies,

$$u(t) = c\left| f_\epsilon(\Psi(t)) \right|^\alpha f_\epsilon(\Psi(t)), \quad c > 0, \alpha \geq 0.$$

Well-posedness

Existence and uniqueness of the solution to the closed-loop system :

classical arguments applying Banach fixed point theorem on finite time intervals and where the lengthes of the time intervals do not depend on the initial state.

Weak ω-limit set

- $\|\Psi(t,.)\|_{L^2(\mathbb{R}^3)} = 1 \quad \Rightarrow \quad \exists t_n \nearrow \infty : \Psi(t_n) \overset{L^2-\text{weak}}{\rightharpoonup} \Psi_0^\infty$.
- $\mathcal{V}_\epsilon(\Psi)$ is continuous with respect to Ψ in the L^2-weak topology.
- In particular

$$\mathcal{V}_\epsilon(\Psi_\infty) = \lim_{n\to\infty} \mathcal{V}_\epsilon(\Psi(t_n)) \leq \mathcal{V}_\epsilon(\Psi_0) < \epsilon,$$

  by the assumptions **A1** and **A2**.
- Is the flow continuous w.r.t initial state in a weak topology ?

Weak ω-limit set : continuity of flow

Define the semi-norm

$$\|\psi\|_{\mathcal{H}} = \max(\|\psi\|_{L^2_{\mu^2}}, \|\psi\|_{L^2_d}) = \max(\|\mu\psi\|_{L^2}, \|\Pi_d\psi\|_{L^2}).$$

Lemma (continuity of flow)

Assume $\mu \in L^p \cap L^\infty(\mathbb{R}^N)$ for some $p \geq 2$. Take α in the feedback law as follows :

$$\begin{cases} \alpha = \frac{p-2N+\varpi(p-N)}{N-\varpi(p-N)}, \quad 0 < \varpi < \frac{N}{N-p}, & \text{if} \quad p \geq 2N, \\ \alpha = 0 & \text{else.} \end{cases}$$

Let $(\Psi_0^n)_{n\in\mathbb{N}}$ be a sequence of $\mathbb{S}$ and $\Psi_0^\infty \in L^2$ with

$$\Psi_0^n \overset{L^2-\text{weak}}{\rightharpoonup} \Psi_0^\infty.$$

Let Ψ_n (resp. $\tilde{\Psi}$) be the weak solutions of the system with $\Psi_n(0) = \Psi_n^0$ (resp. with $\tilde{\Psi}(0) = \pi_d \Psi_0$). Then for every $\tau > 0$,

$$\lim_{n\to\infty} \Psi_n(\tau) = \tilde{\Psi}(\tau) \text{ strongly in } \mathcal{H}.$$

Weak ω-limit set

- Consider :

$$i\frac{d}{dt}\Psi_n = -\triangle\Psi_n + V(x)\Psi_n + u_\epsilon(\Psi_n)\mu(x)\Psi_n, \qquad \Psi_n|_{t=0} = \Psi(t_n),$$
$$i\frac{d}{dt}\widetilde{\Psi} = -\triangle\widetilde{\Psi} + V(x)\widetilde{\Psi} + u_\epsilon(\widetilde{\Psi})\mu(x)\widetilde{\Psi}, \qquad \widetilde{\Psi}|_{t=0} = \widetilde{\Psi}_0 = \Pi_d\Psi_\infty.$$

  Applying the continuity of flow and the continuity of $\mathcal{V}_\epsilon$ with respect to the semi-norm $\mathcal{H}$

$$\mathcal{V}_\epsilon(\Psi_n(\tau)) \rightarrow \mathcal{V}_\epsilon(\widetilde{\Psi}(\tau)), \qquad \text{as } n \rightarrow \infty.$$

- We can therefore show, following the usual steps in the LaSalle's invariance principle, and applying the assumption **A3** and **A4**, that the ω-limit set is included in

$$\{\beta\phi_j \mid \beta \in \mathbb{C}, |\beta| \leq 1, j = 1, ..., N\}$$

Lost mass

- If $\widetilde{\Psi}_0 = \beta\phi_j$ with $|\beta| \leq 1$ and $j \neq 0$ then :

  $$\mathcal{V}_\epsilon(\Psi_\infty) = \mathcal{V}_\epsilon(\widetilde{\Psi}_0) = 1 - (1-\epsilon)|\beta|^2 \geq \epsilon, \quad \Rightarrow \text{ contradiction.}$$

  Thus

  $$\widetilde{\Psi}_0 = \beta\phi_0 \text{ with } 1 - |\beta_0|^2 < \epsilon.$$

  This simply implies that :

  $$\liminf_{t\to\infty} |\langle \Psi(\tilde{t}_n), \phi_0 \rangle|^2 > 1 - \epsilon.$$

Further remarks

Relaxations

The assumptions **A2**,**A3** and **A4** can be relaxed exactly as in the finite dimensional case (Beauchard, Coron, Mirrahimi, Rouchon, System and Control Letters : 2007).

Assumption **A1**

Applying quantum adiabatic theory, it can be relaxed as well,

$$\longrightarrow$$

Conjecture : Assume μ of compact support. For any $\Psi_0 \in L^2$ with $\text{supp}(\Psi_0) \subset \text{supp}(\mu)$ and for any $\epsilon > 0$, we can construct a control law u_ϵ permitting us to reach an ϵ-neighborhood of ϕ_0.

Work in progress !

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1D infinite potential well

$$
\begin{aligned}
i\frac{\partial}{\partial x}\Psi(t,x) &= -\frac{1}{2}\frac{\partial^2\Psi}{\partial x^2}(t,x) - u(t)x\Psi(t,x), \qquad x\in(-\frac{1}{2},\frac{1}{2})\\
\Psi(0,x) &= \Psi_0(x),\\
\Psi(t,\pm\frac{1}{2}) &= 0.
\end{aligned}
$$

Spectra of $A_0 = -\frac{1}{2}\frac{\partial^2}{\partial x^2}$:

$$
\lambda_k = \lambda_{k,0} = \frac{k^2\pi^2}{2}, \quad \phi_k = \phi_{k,0} = \begin{cases} \sqrt{2}\cos(k\pi x), & \text{when } k \text{ is odd,}\\ \sqrt{2}\sin(k\pi x), & \text{when } k \text{ is even.}\end{cases}
$$

1D infinite potential well

Stabilization of the ground state ϕ_1

Main obstacle

The pre-compactness of trajectories in L^2. **We can have phenomena such as the L^2-Mass lost through high-energy levels.**

Main result

Theorem (approximate stabilization)

Let $N \in \mathbb{N}^*$. There exists $\sigma^\sharp = \sigma^\sharp(N) > 0$ such that, for every $\sigma \in (-\sigma^\sharp, \sigma^\sharp) - \{0\}$, $\gamma \in (0,1)$, $\epsilon > 0$, and $\Psi_0 \in \mathbb{S} \cap H^2 \cap H^1_0$ verifying

$$\sum_{k=N+1}^{\infty} |\langle \Psi_0, \phi_{k,\sigma}|^2 < \frac{\epsilon\gamma^2}{1-\epsilon} \quad \text{and} \quad |\langle \Psi_0, \phi_{1,\sigma}\rangle| \geq \gamma,$$

the Cauchy problem

$$i\frac{\partial}{\partial x}\Psi(t,x) = -\frac{1}{2}\frac{\partial^2\Psi}{\partial x^2}(t,x) - u(t)x\Psi(t,x), \qquad x \in (-\frac{1}{2}, \frac{1}{2})$$

$$\Psi(0,x) = \Psi_0(x), \qquad \Psi(t, \pm\frac{1}{2}) = 0,$$

$$u(t) = \sigma + v_{\sigma,N,\epsilon} = \sigma + \Im\left((1-\epsilon)\sum_{k=1}^{N}\left\langle x\Psi, \phi_{k,\sigma}\right\rangle\left\langle \phi_{k,\sigma}, \Psi\right\rangle + \epsilon\left\langle x\Psi, \phi_{1,\sigma}\right\rangle\left\langle \phi_{1,\sigma}, \Psi\right\rangle\right)$$

has a unique strong solution Ψ, moreover, this solution satisfies

$$\liminf_{t\to+\infty} |\langle \Psi(t), \phi_{1,\sigma}\rangle|^2 \geq 1-\epsilon.$$

Idea of the proof

Lyapunov function

$$\mathcal{V}_\epsilon(\Psi) = 1 - (1-\epsilon)\sum_{i=1}^{M}(\Psi_0)|\langle \Psi, \phi_i \rangle|^2 - \epsilon|\langle \Psi, \phi_1 \rangle|^2,$$

$\Psi_0 \in H_0^1 \cap H^2$, $M(\Psi_0)$ is the number of the bound states needed to cover a $1 - O(\epsilon)$ population of Ψ_0.

Continuity of the flow after some technicalities one can prove that the flow of the closed-loop system is continuous for the H^{-1}-norm. Note moreover that $\mathcal{V}_\epsilon$ is continuous for this norm.

Lost mass A population of order $1 - O(\epsilon)$ of the initial state Ψ_0 being covered by the $M(\Psi_0)$ first bound states, the Lyapunov function $\mathcal{V}_\epsilon$ does not allow a mass lost of an order more that $O(\epsilon)$.