Carleman estimates and null controllability for parabolic problems in some non standard cases

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The null controllability problem for the heat equation

$T > 0$
$\Omega \subset \mathbb{R}^N$ bounded
$\emptyset \neq \omega \subset \Omega$

$Q_T := (0, T) \times \Omega$
$\Sigma_T := (0, T) \times \partial\Omega$

The control problem

$$\begin{cases} u_t - \Delta u = h\chi_\omega & Q_T \\ u = 0 & \Sigma_T \\ u(t=0) = u_0 & \Omega \end{cases}$$

Question : null controllability (N.C.) in time T ?

$\forall u_0 \in L^2(\Omega)$, $\exists ? h \in L^2(Q_T)$ such that $u(t = T) \equiv 0$ in Ω ?

I. Standard control theory for parabolic problems

The null controllability problem for the heat equation

Result for the heat equation in bounded domain :

N.C. in any time $T > 0$

- Russell (1973)
  wave eq. E.C. $\rightsquigarrow$ heat eq. N.C., geometric condition on the support of the control, (harmonic analysis, Fourier transform)
- Fattorini/Russell (1971,1974)
  $1 - d$ or $\Omega =$ a ball, (Fourier series, moment problems)
- Lebeau/Robbiano (1995)
  heat equation, constant coeff., distributed or boundary control, (Fourier decomposition, exponential estimate on the size of the control required to control low frequencies, Carleman estimates for elliptic problems)
- Fursikov/Imanuvilov (1995, 1996)
  parabolic equation, variable coeff., (reduction to observability, global Carleman estimates for parabolic problems)

Adjoint problem :

$$\begin{cases} v_t + \Delta v = 0 & Q_T \\ v = 0 & \Sigma_T \end{cases}$$

Observability inequality

$$\int_\Omega v(0)^2 \, dx \leq C_T \int_0^T \int_\omega v^2 \, dxdt$$

Proposition

Observability (in time T) for the adjoint problem
$\Rightarrow$ null controllability (in time T)

- Penalized minimization problem

$$\underset{h\in L^2(\Omega)}{\text{Min}} \left\{ \frac{1}{2}\int_0^T \int_\Omega h^2\, dxdt + \frac{1}{2\varepsilon}\int_\Omega u(T)^2\, dx \right\}$$

existence of a unique minimizer

- Characterization of the minimizer : $\forall \varepsilon > 0$, $h^\varepsilon = -v^\varepsilon \chi_\omega$

$$\begin{cases} v_t^\varepsilon + \Delta v^\varepsilon = 0 & Q_T \\ v^\varepsilon = 0 & \Sigma_T \\ v^\varepsilon(T) = \frac{1}{\varepsilon} u^\varepsilon(T) & \Omega \end{cases}$$

- A priori estimates

$$
\begin{aligned}
\frac{1}{\varepsilon}\int_\Omega u^\varepsilon(T)^2 + \int_0^T\int_\omega (h^\varepsilon)^2 &= \int_\Omega u_0 v^\varepsilon(0) \\
&\leq C_\eta \int_\Omega u_0^2 + \eta \int_\Omega v^\varepsilon(0)^2 \\
&\leq C_\eta \int_\Omega u_0^2 + \eta C \int_0^T \int_\omega (v^\varepsilon)^2 \\
&\leq C \int_\Omega u_0^2 + \frac{1}{2}\int_0^T\int_\omega (v^\varepsilon)^2
\end{aligned}
$$

- Consequence

$$
\frac{1}{\varepsilon}\int_\Omega u^\varepsilon(T)^2 + \frac{1}{2}\int_0^T\int_\omega (h^\varepsilon)^2 \leq C\int_\Omega u_0^2
$$

$$
\Rightarrow \qquad \frac{1}{\sqrt{\varepsilon}} u^\varepsilon(T) \underset{\varepsilon\to 0}{\rightharpoonup} w_T, \quad h^\varepsilon \underset{\varepsilon\to 0}{\rightharpoonup} h \ \text{ and } \ u^\varepsilon \underset{\varepsilon\to 0}{\rightharpoonup} u
$$

- Limit as $\varepsilon \to 0$

$$\begin{cases} u_t^\varepsilon - \Delta u^\varepsilon = h^\varepsilon \chi_\omega & Q_T \\ u^\varepsilon = 0 & \Sigma_T \\ u^\varepsilon(0) = u_0 & \Omega \end{cases} \underset{\varepsilon \to 0}{\longrightarrow} \begin{cases} u_t - \Delta u = h \chi_\omega & Q_T \\ u = 0 & \Sigma_T \\ u(0) = u_0 & \Omega \end{cases}$$

$$\frac{1}{\sqrt{\varepsilon}} u^\varepsilon(T) \underset{\varepsilon \to 0}{\rightharpoonup} w_T \text{ in } L^2(\Omega) \Rightarrow u^\varepsilon(T) \underset{\varepsilon \to 0}{\rightharpoonup} 0$$

$\Rightarrow u(T) = 0$ in Ω $\square$

Question : how to get the observability inequality ?

$$\int_\Omega v(0)^2 \, dx \leq C_T \int_0^T \int_\omega v^2 \, dxdt$$

I. Standard control theory for parabolic problems

Global Carleman estimates

$$\begin{cases} v_t + \Delta v = 0 & Q_T \\ v = 0 & \Sigma_T \end{cases}$$

Global Carleman estimates — **Fursikov/Imanuvilov, 1996**

For S, R large enough,

$$\int_0^T \int_\Omega \left(R^3 S^3 \theta^3 v^2 + RS\theta |\nabla v|^2 \right) e^{-2R\sigma} \, dxdt \leq C_T \int_0^T \int_\omega v^2 \, dxdt$$

$$\sigma(t,x) = \theta(t)(e^{2S\|\Phi\|_\infty} - e^{S\Phi(x)}) > 0 \text{ in } (0,T) \times \Omega$$

$$\theta(t) \to +\infty \text{ as } t \to 0^+, T^-, \qquad \Phi > 0 \text{ in } \Omega$$

$$\Phi_{|\partial\Omega} = 0, \qquad \partial_\nu \Phi_{|\partial\Omega} \leq 0, \qquad \{x \mid \nabla\Phi(x) = 0\} \subset \omega$$

I. Standard control theory for parabolic problems

Application to observability

- Apply Carleman estimates (zero order term) :

$$\int_0^T \int_\Omega v^2 \underbrace{\theta^3 e^{-2R\sigma}}_{\to 0 \text{ as } t \to 0^+, T^-} \, dxdt \leq C \int_0^T \int_\omega v^2 \, dxdt$$

$$\Rightarrow \int_{T/4}^{3T/4} \int_\Omega v^2 \, dxdt \leq C \int_0^T \int_\omega v^2 \, dxdt$$

- Use $t \mapsto \int_\Omega v(t)^2 \, dx$ increasing :

$$\int_\Omega v(0)^2 \, dx \leq \frac{2}{T} \int_{T/4}^{3T/4} \int_\Omega v^2 \, dxdt \leq C \int_0^T \int_\omega v^2 \, dxdt$$

I. Standard control theory for parabolic problems

Global Carleman estimates = powerful tool

- Second order parabolic operators with variable coefficients
  $\Omega \subset \mathbb{R}^N$, T>0 $\quad A : \Omega \to \mathcal{S}_n(\mathbb{R})$ (symmetric matrices)
  $\emptyset \neq \omega \subset \Omega \quad V : \Omega \to \mathbb{R}$

Control problem

$$\begin{cases} u_t - \operatorname{div}(A(x)\nabla u) + V(x)u = h\chi_\omega & Q_T \\ u = 0 & \Sigma_T \\ u(t=0) = u_0 & \Omega \end{cases}$$

Fursikov/Imanuvilov (Lect. Notes, 1996)

Under *"standard"* assumptions : global Carleman estimates
$\Rightarrow$ observability inequality $\Rightarrow$ N.C. in any time $T > 0$

- Application to more general operators and to semi-linear problems

I. Standard control theory for parabolic problems

Part of the other references on the subject

- Carleman estimates and appl. to control and UCP : Tataru (94-97...), Albano/Tataru (00,04)...
- Semi-linear problems, (approx. C., N.C. and finite dim. N.C.) : Fabre/Puel/Zuazua (95), Fernàndez-Cara (97), Zuazua (97), Fernàndez-Cara/Zuazua (00), Aniţa/Tataru (02), Coron/Trélat (04,06)...
- Navier-Stokes equations : Fursikov/Imanuvilov (94,96,99), Coron (95), Coron-Fursikov (96), Imanuvilov (97,98,01), Barbu (02), Fernàndez-Cara/Guerrero/Imanuvilov/Puel (04,05,06), Barbu/Lasiecka/Triggiani (stabilization, 06,07), Raymond (stabilization, 06,07)...
- Hyperbolic Carleman estimates : Lasiecka/Triggiani/Yao (97), X. Zhang (00), Imanuvilov/Yamamoto (03)...
- Schrödinger equation : Baudouin/Puel (02), Lasiecka/Triggiani/Zhang (04), Mercado/Osses/Rosier (08)...

I. Standard control theory for parabolic problems

Field of application of the standard theory

"Standard" assumptions for global Carleman estimates :

1. Ω is a bounded domain
2. A regular (smooth coeff.) + uniform ellipticity condition :
   $\exists \alpha > 0 \qquad \forall x \in \overline{\Omega} \qquad \forall \xi \in \mathbb{R}^N \qquad A(x)\xi \cdot \xi \geq \alpha |\xi|^2$
   ($\rightsquigarrow$ uniformly parabolic equation)
3. V is regular "enough"
   $V \in L^\infty(\Omega)$ or even $V \in L^p(\Omega)$ with $p > 2N/3$

References : - Fursikov/Imanuvilov (Lect. Notes, 1996)
- Imanuvilov/Yamamoto (2003)

Question : what happens without one of the above assumptions ?

Objectives of this talk

Discuss 3 *"non-standard"* situations

1. Ω unbounded
2. A regular elliptic but non uniformly elliptic,
   i.e. A positive definite in Ω but not in $\overline{\Omega}$
   ex : vanishing diffusion coefficient at the boundary in $1-d$
3. V singular $\notin L^p(\Omega)$ with $p > 2N/3$
   ex : inverse-square potential $V(x) = \dfrac{\lambda}{|x|^2}$

Discuss the links between these 3 situations

$$1 \underset{\text{strong degeneracy}}{\longleftarrow} 2 \underset{\text{weak degeneracy}}{\longrightarrow} 3$$

Objectives of this talk

Motivations

- Degenerate parabolic problems
  - arise in many domains : physics, biology, economy..
  - A definite positive only in Ω : gene frequency models (Wright-Fischer, Fleming-Viot) ; Budyko-Sellers climate models ; Black-Scholes (finance theory)
  - A non definite positive in Ω : transport/diffusion coupling
    - linearized Crocco eq. (boundary layer model) : Raymond/Martinez/V. (SICON, 2003)
    - Mc Kendrick age-structured population dynamics models : Ainseba/Anița (2001,2004), Ainseba/Iannelli (2003)...
  - p-Laplacian degeneracy : porous media, non-Newtonian flows
- Inverse-square potentials
  - quantum mechanics (Baras/Goldstein, 1994)
  - linearized combustion models around singular stationary solutions : Mignot/Puel (1988), Brézis/Vázquez (1997)
- Other non standard situations
  - discontinuous coefficients (stratified media)
    - Le Rousseau (07), Benabdallah/Demerjian/Le Rousseau (07)

II. Parabolic problems in unbounded domains

Negative N.C. results when ω bounded

The heat equation in the half line : $\Omega = (0, +\infty)$, $V \in L^\infty(\Omega)$

$$\begin{cases} u_t - u_{xx} + V(x)u = h\chi_\omega & x \in (0, +\infty),\ t > 0 \\ u(x = 0) = 0 & t > 0 \\ u(t = 0) = u_0 & x \in (0, +\infty) \end{cases}$$

Escauriaza/Seregin/Sverak (2003,2004) : backward uniqueness

$$\omega = (0, R) \text{ and } u(t = T) \equiv 0 \Rightarrow \text{supp}\,(u_0) \subset \omega$$

Consequence : N.C. is false when ω bounded

Previous result : Micu/Zuazua (2001,2002) : $V = 0$, boundary control at $x = 0 \Rightarrow$ N.C. is false even for I.C. in $\mathcal{D}(\Omega)$

Conclusion

Ω unbounded + ω bounded $\Rightarrow$ N.C. false

(even in L^2-weighted spaces)

II. Parabolic problems in unbounded domains

Positive N.C. results when ω unbounded

- Cabanillas/De Menezes/Zuazua (2001) :
  $\omega = (\alpha, +\infty) \Rightarrow$ N.C. holds (in L^2)
  (semi-linear problem in dim N, hyp : $\Omega \setminus \omega$ bounded)
  Question : smaller (unbounded) ω ?

- Cannarsa/Martinez/V. (COCV, 2004)
  $\omega = \cup_{n\in\mathbb{N}} I_n \Rightarrow$ N.C. in suitable L^2-weighted spaces
  Example 1 : $|I_n| = cste \Rightarrow$ N.C.in L^2
  Example 2 : $|\omega| < +\infty$
  - N.C. holds in suitable L^2-weighted spaces :
    $\forall u_0 \in L^2_{1/\rho_1}(\Omega),\ \exists h \in L^2(0,T;L^2(\Omega))$ s.t. $u(T) \equiv 0$
    $\forall u_0 \in L^2(\Omega),\ \exists h \in L^2(0,T;L^2_{\rho_2}(\Omega))$ s.t. $u(T) \equiv 0$
  - N.C. in L^2 is false (Miller, 2004)

  Proof : cut-off arguments (work in bounded regions) + standard Carleman estimates (sharp estimates of the constants with respect to $(I_n)_n \rightsquigarrow \rho_1, \rho_2$)

Recall : ω bounded $\Rightarrow$ N.C. false (even in L^2-weighted spaces)

Question : replace N.C. by a weaker notion ?

Regional null controllability (R.N.C.) in time T

$\forall u_0 \in L^2(\Omega),\ \exists ? h \in L^2(Q_T)$ such that $u(t = T) \equiv 0$ in D where D is a well-chosen sub-domain of Ω ?

Questions :

- find the sub-domain D ?
- required observability inequality ?
- associated penalized problem ?

II. Parabolic problems in unbounded domains

Positive weaker N.C. results when ω bounded

The heat equation in the half line : $\Omega = (0, +\infty)$, $\omega = (\alpha, \beta)$

$$\begin{cases} u_t - u_{xx} = h\chi_{(\alpha,\beta)} & x \in (0,+\infty),\ t > 0 \\ u(x=0) = 0 & t > 0 \\ u(t=0) = u_0 & x \in (0,+\infty) \end{cases}$$

Theorem Cannarsa/Martinez/V. (CPAA, 2002)

$\forall T > 0,\ \forall \delta > 0,\ \forall u_0 \in L^2(\Omega)$,
$\exists h \in L^2(Q_T)$ such that $u(t = T) \equiv 0$ in $D = (0, \beta - \delta)$.

Remark : $D = (0, \beta - \delta)$ is optimal :
$\forall \delta > 0$, R.N.C. in $D' = (0, \beta + \delta)$ is false

II. Parabolic problems in unbounded domains

Proof of R.N.C. on $D = (0, \beta - \delta)$

- Modified penalized problems

$$\underset{h \in L^2(\Omega)}{\text{Min}} \left\{ \frac{1}{2} \int_0^T \int_\Omega h^2 \, dxdt + \frac{1}{2\varepsilon} \int_D u(T)^2 \, dx \right\}$$

where D replaces Ω

- Characterization of the minimizer : $\forall \varepsilon > 0, \;\; h^\varepsilon = -v^\varepsilon \chi_\omega$

$$\begin{cases} v_t^\varepsilon + \Delta v^\varepsilon = 0 & Q_T \\ v^\varepsilon = 0 & \Sigma_T \\ v^\varepsilon(T) = \frac{1}{\varepsilon} u^\varepsilon(T) \chi_D & \Omega \end{cases}$$

II. Parabolic problems in unbounded domains

Proof of R.N.C. on $D = (0, \beta - \delta)$

- A priori estimates

$$\frac{1}{\varepsilon}\int_D u^\varepsilon(T)^2 + \int_0^T \int_\omega (h^\varepsilon)^2 \le C_\eta \int_\Omega u_0^2 + \eta \int_\Omega v^\varepsilon(0)^2$$

$$\underset{???}{\le} C_\eta \int_\Omega u_0^2 + \eta C \int_0^T \int_\omega (v^\varepsilon)^2$$

$$\Rightarrow \qquad \frac{1}{\varepsilon}\int_D u^\varepsilon(T)^2 + \frac{1}{2}\int_0^T \int_\omega (h^\varepsilon)^2 \le C \int_\Omega u_0^2$$

- Limit as $\varepsilon \to 0$

$h^\varepsilon \underset{\varepsilon\to 0}{\rightharpoonup} h,\ u^\varepsilon \underset{\varepsilon\to 0}{\rightharpoonup} u,$ $\qquad u,\ h$ solution of the control problem

$$\frac{1}{\sqrt{\varepsilon}} u^\varepsilon(T)\chi_D \underset{\varepsilon\to 0}{\rightharpoonup} w_T \text{ in } L^2(\Omega) \Rightarrow u^\varepsilon(T)\chi_D \underset{\varepsilon\to 0}{\rightharpoonup} 0 \Rightarrow u(T) = 0 \text{ in } D \quad \square$$

II. Parabolic problems in unbounded domains

The required observability inequality

$$\begin{cases} v_t + \Delta v = 0 & Q_T \\ v = 0 & \Sigma_T \end{cases}$$

Observability inequality for R.N.C. on D

$$\int_\Omega v(0)^2\,dx \leq C_T \int_0^T \int_\omega v^2\,dxdt + C_T \int_{\Omega\setminus D} v(T)^2\,dx$$

Application : $v^\varepsilon(T) = \frac{1}{\varepsilon} u^\varepsilon(T)\chi_D = 0$ in $\Omega \setminus D$

Hence $\displaystyle\int_\Omega v^\varepsilon(0)^2\,dx \leq C_T \int_0^T \int_\omega (v^\varepsilon)^2\,dxdt$

Proof : cut-off argument $\rightsquigarrow$ split the domain into two parts

$\rightarrow$ $(0, \beta-\delta)$ bounded : use standard Carleman estimates

$\rightarrow$ $(\beta-\delta, +\infty)$ unbounded : use energy inequality

III. Degenerate parabolic problems

Effect of a vanishing diffusion coefficient at the boundary ?

A class of 1-dimensional degenerate heat equations

$$\begin{cases} u_t - (a(x)u_x)_x = h\chi_\omega & x \in (0,1),\ t > 0 \\ \text{B.C. at } x = 0, 1 & t > 0 \\ u(0) = u_0 & x \in (0,1) \end{cases}$$

Typical example

$a(x) = x^\alpha, \qquad \alpha \geq 0 \qquad \rightsquigarrow \qquad$ difficulty : $a(0) = 0$

B.C. at $x = 1$: $u(t,1) = 0$ (Dirichlet)

B.C. at $x = 0$: $\begin{cases} u(t,0) = 0 & \text{if } \alpha < 1 \\ (au)_x(t,0) = 0 & \text{if } \alpha \geq 1 \end{cases}$

III. Degenerate parabolic problems

Well-posedness in suitable weighted Sobolev spaces

Campiti/Metafune/Pallara (Semigroup Forum, 1998)

- **Weakly degenerate case $\alpha < 1$** :
  $H^1_\alpha(0,1) = \{u \in L^2(0,1) \mid \int_0^1 x^\alpha u_x^2\, dx < \infty,$
  $u(0) = 0 = u(1)\}$
  $D(\mathcal{A}) = \{u \in H^1_\alpha(0,1) \mid x^\alpha u_x \in H^1(0,1)\}$ $\quad \mathcal{A}u = (x^\alpha u_x)_x$

  • $u_0 \in L^2(0,1), \quad h \in L^2(Q_T)$
  $\rightsquigarrow\; u \in \mathcal{C}([0,T]; L^2(0,1)) \cap L^2(0,T; H^1_\alpha(0,1))$

  • $u_0 \in H^1_\alpha(0,1), \quad h \in L^2(Q_T)$
  $\rightsquigarrow\; u \in \mathcal{C}([0,T]; H^1_\alpha(0,1)) \cap H^1(0,T; L^2(0,1)) \cap L^2(0,T; D(A))$

- **Strongly degenerate case $\alpha \geq 1$** : similar results
  B.C. $(au)_x(0) = 0$ included in $D(\mathcal{A})$

III. Degenerate parabolic problems

Role of the "degree" of degeneracy

Change of variables : $X = \int_x^1 \frac{ds}{\sqrt{a(s)}}$ $\quad U(t,X) = a(x)^{1/4} u(t,x)$

$$\begin{cases} u_t - (au_x)_x = h\chi_\omega \\ x \in (0,1) \end{cases} \quad \rightsquigarrow \quad \begin{cases} U_t - U_{XX} + V(X)U = \tilde{h}\chi_{\tilde{\omega}} \\ X \in \tilde{\Omega} \end{cases}$$

$\alpha \geq 2$	$V \in L^\infty(\tilde{\Omega})$	$\tilde{\Omega}$ unbounded
$\alpha < 2$ $(\alpha = 4/3)$	V singular $(V = 0)$	$\tilde{\Omega}$ bounded

$\Rightarrow$ 2 cases where standard Carleman estimates do not apply :

- strong degeneracy ($\alpha \geq 2$) :
  use the results for unbounded domains
- weak degeneracy ($\alpha < 2$) :
  Remark : $\alpha = 4/3 \Rightarrow$ N.C. is true (since $V = 0$)
  Question : N.C. for all $0 \leq \alpha < 2$?
  specific Carleman estimates ?

III. Degenerate parabolic problems

The strongly degenerate case

Effect of the change of variables :

$$0 \rightsquigarrow +\infty \qquad \Omega = (0,1) \rightsquigarrow \tilde{\Omega} = (0,+\infty)$$
$$1 \rightsquigarrow 0 \qquad \omega = (x_0, x_1) \rightsquigarrow \tilde{\omega} = (X_1, X_0)$$

Consequence : $\alpha \geq 2 \Rightarrow$ N.C. is false (in general)
More precisely :

- $x_0 = 0 \Rightarrow \tilde{\omega} = (X_1, +\infty)$ is unbounded
  - N.C. is true
- $x_0 \neq 0 \Rightarrow \tilde{\omega} = (X_1, X_0)$ is bounded
  - N.C. is false
  - R.N.C. is true on $D = (x_0 + \delta, 1)$
  - D is optimal

III. Degenerate parabolic problems

Main result : $0 \leq \alpha < 2 \Rightarrow$ N.C. is true

Theorem — Cannarsa/Martinez/V. (SICON, 2007)

Assume $0 \leq \alpha < 2$. Then N.C. holds : $\forall T > 0$, $\forall (x_0, x_1) \neq \emptyset$, $\forall u_0 \in L^2(\Omega)$, $\exists h \in L^2(Q_T)$ such that $u(T) \equiv 0$ in $(0, 1)$.

Proof :

- show equivalence with observability inequality (classical)
- derive specific Carleman estimates (main step !)
- deduce observability inequality

Major changes needed for degenerate parabolic equations :

- the weights in Carleman estimates must be adapted to degeneracy
- to deduce observability, Carleman estimates have to be used up to the first order term instead of the zero order term
- Hardy's inequalities become essential (both to prove Carleman estimate and to deduce observability)

III. Degenerate parabolic problems

Carleman estimates for weakly degenerate problems

Adjoint problem : $\begin{cases} v_t + (x^\alpha v_x)_x = 0 & Q_T \\ \text{B.C.} & \Sigma_T \end{cases}$

Theorem Cannarsa/Martinez/V. (SICON, 2007)

Assume $0 \le \alpha < 2$. Then, for R large enough,

$$\int_0^T \int_\Omega \left(R^3 \theta^3 x^{2-\alpha} v^2 + R\theta x^\alpha v_x^2 \right) e^{-2R\sigma} \, dxdt \le C_T \int_0^T \int_\omega v^2 \, dxdt$$

where $\sigma(x,t) = \theta(t)(2 - x^{2-\alpha})$

Previous result : Cannarsa/Martinez/V. (Adv. Diff. Eq., 2005)

$$\int_0^T \int_\Omega R^3 \theta^3 v^2 e^{-2R\sigma} \, dxdt \le C \int_0^T \int_\omega v^2 \, dxdt$$

$\rightsquigarrow$ estimates closer to standard Carleman estimates

BUT only proved for $\alpha \in [0, 1/2]$ or $\alpha \in [5/4, 2)$!

III. Degenerate parabolic problems

Observability for weakly degenerate problem

Implication : Carleman estimates $\Rightarrow$ observability ?

$$\int_0^T \int_\Omega R^3 x^{2-\alpha} v^2 \underbrace{\theta^3 e^{-2R\sigma}}_{\to 0 \text{ as } t \to 0^+, T^-} + R x^\alpha v_x^2 \underbrace{\theta e^{-2R\sigma}}_{\to 0 \text{ as } t \to 0^+, T^-} \leq C_T \int_0^T \int_\omega v^2$$

$$\Rightarrow \int_{T/4}^{3T/4} \int_\Omega \Big(R^3 \underbrace{x^{2-\alpha}}_{\to 0 \text{ as } x \to 0} v^2 + R x^\alpha v_x^2 \Big) \, dxdt \leq C_T \int_0^T \int_\omega v^2 \, dxdt$$

Remarks :

- zero order estimate is too weak !
- must use gradient estimate + Hardy inequalities : $\alpha \in [0, 2) \setminus \{1\}$

$$\forall \varphi \in H^1_\alpha(0,1), \qquad \int_0^1 \frac{\varphi^2}{x^{2-\alpha}} \, dx \leq \frac{4}{(\alpha - 1)^2} \int_0^1 x^\alpha \varphi_x^2 \, dx$$

III. Degenerate parabolic problems

Proof : Carleman estimates ⇒ observability

- Apply Carleman estimates (first order term) :

$$\int_{T/4}^{3T/4} \int_\Omega x^\alpha v_x^2 \, dxdt \leq C \int_0^T \int_\omega v^2 \, dxdt$$

- Use $t \mapsto \int_0^1 x^\alpha v_x(t)^2 \, dx$ increasing :

$$\int_\Omega x^\alpha v_x(0)^2 \, dx \leq \frac{2}{T} \int_{T/4}^{3T/4} \int_\Omega x^\alpha v_x^2 \, dxdt \leq C \int_0^T \int_\omega v^2 \, dxdt$$

- Use Hardy inequalities :

$$\int_0^1 v^2(0) \, dx \leq \int_0^1 \frac{v^2(0)}{x^{2-\alpha}} \, dx \leq C \int_0^1 x^\alpha v_x^2(0) \, dx \leq C \int_0^T \int_\omega v^2 \, dxdt$$

III. Degenerate parabolic problems

Extensions to more general weakly degenerate problems

- Martinez/V. (J. Evol. Eq., 2006)
  general coefficient $a(x)$ with $1/\sqrt{a} \in L^1(0,1)$:
  $u_t - (a(x)u_x)_x = \chi_\omega h$
- Alabau-Boussouira/Fragnelli/Cannarsa (J. Evol. Eq., 2006)
  semi-linear problem + general coefficient $a(x)$:
  $u_t - (a(x)u_x)_x + g(u) = \chi_\omega h$
- Cannarsa/Fragnelli/Rocchetti (Netw. Heterog. Media 2007)
  non-divergence form + drift term :
  $u_t - a(x)u_{xx} + b(x)u_x = \chi_\omega h$
- Cannarsa/Martinez/V. (work in progress)
  higher dimension operator with degeneracy at the boundary in the normal direction : $u_t - \operatorname{div}(A(x)\nabla u) = \chi_\omega h$
  e-values of $A(x) = \{\lambda_1(x), \lambda_2(x)\}$ with e-vectors $\varepsilon_1(x), \varepsilon_2(x)$

$$\begin{cases} \lambda_1(x) = d_{\partial\Omega}(x)^\alpha , \quad \varepsilon_1(x) = \nu(x) & \text{near } \partial\Omega \\ \lambda_2(x) \geq m > 0 & \forall x \in \Omega \end{cases}$$

IV. Parabolic problems with an inverse-square potential

Replace $-\Delta$ by $-\Delta - V(x)I$ in the heat equation ?

$T > 0$ $\quad 0 \in \Omega \subset \mathbb{R}^N$ bounded $\quad \emptyset \neq \omega \subset \Omega$

$$\begin{cases} u_t - \Delta u - V(x)u = h\chi_\omega & Q_T \\ u = 0 & \Sigma_T \\ u(t=0) = u_0 & \Omega \end{cases}$$

- $V \in L^\infty(\Omega)$: standard results holds (well-posedness + N.C)
- Question : inverse square potentials $V(x) = \lambda/|x|^2$?

Essential tool : Hardy inequality

$$\forall \varphi \in H_0^1(\Omega), \qquad \underbrace{\frac{(N-2)^2}{4}}_{=:\lambda_\star(N)} \int_\Omega \frac{\varphi^2}{|x|^2}, dx \leq \int_\Omega |\nabla \varphi|^2 \, dx$$

$\Rightarrow -\Delta - \lambda|x|^{-2}I$ still coercive in $H_0^1(\Omega)$ when $\lambda < \lambda_\star(N)$
and at least nonnegative in $H_0^1(\Omega)$ when $\lambda \leq \lambda_\star(N)$

IV. Parabolic problems with an inverse-square potential

Well-posedness

Baras/Goldstein (Trans. AMS, 1984)
Vázquez/Zuazua (J. Funct. Anal., 2000)

- **super-critical case : $\boldsymbol{\lambda > \lambda_\star(N)}$**
  lack of well-posedness
  (instantaneous blow-up of positive solutions)
- **sub-critical case : $\boldsymbol{\lambda < \lambda_\star(N)}$**
  global existence in standard functional setting
  $u_0 \in L^2(\Omega), h \in L^2(Q_T)$
  $\Rightarrow u \in \mathcal{C}([0, T]; L^2(\Omega)) \cap L^2(0, T; H_0^1(\Omega))$
- **critical case : $\boldsymbol{\lambda = \lambda_\star(N)}$**
  replace $H_0^1(\Omega)$ by $H := \overline{\mathcal{D}(\Omega)}^{\|\cdot\|_H}$

$$\|\varphi\|_H^2 := \int_\Omega |\nabla \varphi|^2 \, dx - \lambda_\star(N) \int_\Omega \frac{\varphi^2}{|x|^2} \, dx$$

$u_0 \in L^2(\Omega), h \in L^2(Q_T) \Rightarrow u \in \mathcal{C}([0, T]; L^2(\Omega)) \cap L^2(0, T; H)$

Remark : interesting case $\rightsquigarrow 0 \notin \omega$

The $1-d$ case :

$$\begin{cases} u_t - u_{xx} - \dfrac{\lambda}{|x|^2} u = h\chi_\omega & x \in (0,1),\ t > 0 \\ u(x=0) = 0 = u(x=1) & t > 0 \\ u(t=0) = u_0 & x \in (0,1) \end{cases}$$

Theorem : V./Zuazua (J. Funct. Anal., 2008)

Assume $\lambda \leq \lambda_\star(1) = 1/4$. Then N.C. holds.

Proof : specific Carleman estimates $\Rightarrow$ observability $\Rightarrow$ N.C.

IV. Parabolic problems with an inverse-square potential

Carleman estimates for problems with singular potentials

Adjoint problem :

$$\begin{cases} v_t + v_{xx} + \dfrac{\lambda}{|x|^2} v = 0 & x \in (0,1),\ t > 0 \\ v(x=0) = 0 = v(x=1) & t > 0 \end{cases}$$

Theorem : V./Zuazua (J. Funct. Anal., 2008)

Assume $\lambda \leq \lambda_\star(1) = 1/4$. Then, for all $\gamma < 2$,

$$\int_0^T \int_0^1 R^3\theta^3 x^2 v^2 e^{-2R\sigma} + \left(\frac{1}{4} - \lambda\right) \int_0^T \int_0^1 R\theta \frac{v^2}{x^2} e^{-2R\sigma}$$

$$+ \int_0^T \int_0^1 R\theta \frac{v^2}{x^\gamma} e^{-2R\sigma} \leq C_T \int_0^T \int_\omega v^2$$

Application to observability : $\gamma = 0$

$$\int_0^1 v(0)^2\, dx \leq \frac{2}{T} \int_{T/4}^{3T/4} \int_0^1 v^2\, dxdt \leq C_T \int_0^T \int_\omega v^2$$

- **Extension to the $\mathbf{N-d}$ case : N.C. for $\lambda \leq \lambda_\star(\mathbf{N})$**
  - V./Zuazua (J. Funct. Anal, 2008) :
    specific geometric conditions (ω circles the singularity)
    use cut-off arguments + spherical harmonics + $1-d$ estimates
  - Ervedoza (preprint, 2008) :
    no geometric condition
    extension of Carleman estimates to the $N-d$ case

- **Lack of N.C. in the super-critical case $\lambda > \lambda_\star(\mathbf{N})$**
  - Ervedoza (preprint, 2008) :
    regularized potential + sequence of eigenfunctions
    whose energies concentrate around the singularity
    $\rightsquigarrow$ lack of observability (when $0 \notin \omega$)

- **Wave and Schrödinger equations with inverse square potentials**
  - V./Zuazua (preprint, 2008) : boundary control

    $\rightarrow$ N.C. for $\lambda \leq \lambda_\star(N)$

    - multiplier method + specific Hardy type inequality :

$$\int_\Omega (x \cdot \nabla\varphi)^2 \, dx \leq R_\Omega^2 \int_\Omega \left(|\nabla\varphi|^2 - \lambda_\star(N) \frac{\varphi^2}{|x|^2} \right) + \frac{N^2-4}{4} \int_\Omega \varphi^2$$

for all $\varphi \in H_0^1(\Omega)$ and with $R_\Omega := \max_{x\in\Omega} |x|$.

- limit of the method : multiplier centered at the singularity

$\rightsquigarrow$ non expected restriction on the region of control

(e.g., Ω convex $\rightsquigarrow$ control on the *whole* boundary)

$\rightarrow$ Lack of N.C. for $\lambda > \lambda_\star(N)$

  - current work with L. Baudouin :

    - derive hyperbolic Carleman estimates for problems with singular potentials ?
    - weaken the geometric conditions ?

V. Conclusion and perspectives

Initial motivation : linearized Crocco equation

boundary layer model

$\downarrow$

Prandtl's equations $\longleftarrow$ nonlinear degenerate

$\downarrow$ $\longleftarrow$ $\begin{cases} \text{Crocco's transormation} \\ + \text{ linearization} \end{cases}$

Crocco's equation

$$\begin{cases} u_t + b(y)u_x - \big(a(y)u_y\big)_y = 0 & x \in (0, L), y \in (0, 1) \\ + \text{ boundary and initial conditions} \end{cases}$$

- double degeneracy : $A(x, y) = \begin{pmatrix} 0 & 0 \\ 0 & a(0) \end{pmatrix}$ with $a(0) = 0$
- other difficulty : $b(1) = 0$

V. Conclusion and perspectives

Application to the linearized Crocco equation

- Motivated the analysis of 2 simplified models :
  - Martinez/Raymond/V. (2003) :

$$\begin{array}{ccccccc} u_t & + & b(y)u_x & - & (a(y)u_y)_y & = 0 \\ & & \downarrow & & \downarrow & \\ u_t & + & u_x & - & u_{yy} & = 0 \end{array}$$

  - Cannarsa/Martinez/V. (2005, 2007)
    $u_t - (a(y)u_y)_y = 0$ with $a(0) = 0$
- Application : partial results for the Crocco type equation
  $u_t + u_x - (a(y)u_y)_y = 0$ with $a(0) = 0$
- Perspective : $u_t + b(y)u_x - (a(y)u_y)_y = 0$ with $b(1) = 0$
- Other perspectives : Fleming-Viot model, Budyko-Seller model...